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Existence and Approximation of Fixed Points of Enriched φ-Contractions in Banach Spaces

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Abstract: We introduce the class of enriched φ-contractions in Banach spaces as a natural generalization of φ-contractions and study the existence and approximation of the fixed points of mappings in this new class, which is shown to be an unsaturated class of mappings in the setting of a Banach space. We illustrated the usefulness of our fixed point results by studying the existence and uniqueness of the solutions of some second order (p,q)-difference equations with integral boundary value conditions.

Keywords: Banach space; enriched φ-contraction; enriched cyclic φ-contraction; fixed point; Maia type fixed point theorem; (p,q)-difference equation; integral boundary value condition

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1. Introduction

One hundred years ago, in 1922, Banach published his seminal paper [1], where, amongst many other fundamental results, it has been stated the first version of the contraction mapping principle, commonly called the Picard-Banach or Banach fixed point theorem. This was the inception of the metric fixed point theory which afterwards developed in an extraordinary impressive way in many theoretical and applicative directions, see the monographs [2–8] and references therein.

In its original form, the Banach contraction mapping principle was stated in the setting of a complete linear normed space - what we are calling nowadays a Banach space - while its formulation in the more general setting of a metric space is due to Caccioppoli [9]. In its simpler form, it is stated as follows (for a complete statement, including *a priori* error estimate, *a posteriori* error estimate and rate of convergence, see for example [10]).

Theorem 1. *Let (X,d) be a complete metric space and $T : X \to X$ a strict contraction, i.e., a map satisfying*

$$d(Tx,Ty) \le a\, d(x,y)\,, \quad \text{for all } x,y \in X\,, \tag{1}$$

where $0 \le a < 1$ is constant. Then:

$(p1)$ T has a unique fixed point p in X (i.e., $Tp = p$);

$(p2)$ The Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = Tx_n\,, \quad n = 0,1,2,\ldots \tag{2}$$

converges to p, for any $x_0 \in X$.

Remark 1. *A map satisfying $(p1)$ and $(p2)$ in Theorem 1 is said to be* a Picard operator, *see [8,11–13], for more details.*

Banach fixed point theorem is a simple and useful tool in establishing existence and uniqueness theorems for operator equations. This is the reason why Theorem 1 has a very important role in nonlinear analysis and has motivated researchers to try to extend and generalise it in order to extend its area of applications.

One way to generalize Theorem 1 was to consider more advanced conditions in the inequality while another way is to relax the assumptions concerning the space itself. A third direction is to combine the two previous ways of generalization. Therefore, the amount of papers devoted to these kinds of generalizations and many other variants is enormous.

A class of these generalisations is based on considering only continuous mappings, like in the original case of the contraction mapping in Theorem 1, by replacing condition (1) by a weaker contractive condition of the form

$$d(Tx, Ty) \leq \varphi(d(x, y)), x, y \in X \tag{3}$$

or of the form

$$d(Tx, Ty) \leq \psi(d(x, y))d(x, y), x, y \in X, \tag{4}$$

where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ and $\psi : \mathbb{R}_+ \to [0, 1)$ are functions possessing some suitable properties.

A mapping T satisfying (3) or (4) is commonlly called a *φ-contraction* (*ψ-contraction*). Obviously, a strict contraction is a φ-contraction with $\varphi(t) = at, t \in [0, \infty)$ and $a \in [0, 1)$.

Fixed point theorems for mappings satisfying the contraction condition (3) were first obtained in 1968 by Browder [14] who proved that, if φ is nondecreasing and right continuous, then, in a complete metric space (X, d), any φ-contraction T has a unique fixed point $p \in X$ and

$$\lim_{n\to\infty} T^n x_0 = p$$

for $x_0 \in X$ arbitrary chosen.

In 1969, Boyd and Wong [15] extended Browder's result by weakening the hypothesis on φ, by assuming only that φ is right upper semi-continuous (and is not necessarily monotonic).

A few years later, Matkowski [16,17] and Rus [18] extended Boyd and Wong's results by considering φ-contractions with φ a so called *comparison function* (see [8] for the definition, properties, and more details on this concept as well as a comprehensive list of references).

If we refer now to the fixed point theorems for mappings satisfying the contraction condition (4), we note the result of Rakotch [19] from 1962 who considered ψ-contractions satisfying (4) with ψ nonincreasing and $\psi(t) < 1$ for $t > 0$.

In 1973, Geraghty [20] considered another kind of properties for the altering function $\psi : [0, \infty] \to [0, 1)$ appearing in (4), i.e., he assumed that $\psi(t_n) \to 1$ implies $t_n \to 0$.

The fixed point theorems for φ-contractions, apart of their theoretical importance, are also important from the point of view of applications, as a contraction condition of the form (3) or (4) is naturally derived from many operator equations. We exemplify this by some very recent papers that use fixed point theorems for φ-contractions to the study of: nonlinear integral equations [21]; fractional differential equations with nonlocal multi-point boundary conditions [22]; boundary value problems for Hilfer fractional differential equations [23,24]; nonlinear Volterra integral equations [25]; boundary value problems for second-order (p, q)-difference equations [26] etc.

On the other hand, Berinde and Păcurar [27] extended the contraction mapping principle, in the setting of a Banach space, to the so called class of *enriched contractions*. It has been also shown in [27] that any Banach contraction T is an enriched contraction (but not the reverse) and that some nonexpansive mappings are enriched contractions.

Starting from the facts presented above, our aim in this paper is to introduce the class of enriched φ-contractions as a unifying concept of φ-contractions and enriched contractions and study them from the point of view of the existence and approximation of their fixed points. Such an approach will extend significantly the area of applications of the class of φ-contractive type mappings.

Following the terminology and results in [28], we also show that the class of enriched φ-contractions is an *unsaturated* class of mappings in the setting of a Banach space, which means that the enriched φ-contractions are effective generalization of φ-contractions.

2. Boyd-Wong Type Fixed Point Theorems for Enriched φ-Contractions

By definition, $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a *comparison function* (see for example [4]), if the following two conditions hold:

(i_φ) φ is nondecreasing, i.e., $t_1 \le t_2$ implies $\varphi(t_1) \le \varphi(t_2)$.

(ii_φ) $\{\varphi^n(t)\}$ converges to 0 for all $t \ge 0$.

It is obvious that any comparison function also possesses the following property:

(iii_φ) $\varphi(t) < t$, for $t > 0$.

Prototypes of comparison functions are (see also φ in Example 1):

$$\varphi(t) = \frac{t}{t+1}, t \in [0, \infty);\ \varphi(t) = \frac{t}{2}, t \in [0,1] \text{ and } \varphi(t) = t - \frac{1}{3}, t \in (1, \infty);$$

(one can see that a comparison function is not necessarily continuous)

Definition 1. *Consider a linear normed space $(X, \|\cdot\|)$ and let $T : X \to X$ be a self mapping. T is said to be an enriched φ-contraction if one can find a constant $b \in [0, +\infty)$ and a comparison function φ such that*

$$\|b(x-y) + Tx - Ty\| \le (b+1)\varphi(\|x-y\|), \forall x, y \in X. \tag{5}$$

We shall also call T a (b, φ)-enriched contraction.

Example 1. *1. Any (b, θ)-enriched contraction (see [27]), i.e., any self mapping $T : X \to X$ for which there exist $b \in [0, +\infty)$ and $\theta \in [0, b+1)$ such that*

$$\|b(x-y) + Tx - Ty\| \le \theta\|x-y\|, \forall x, y \in X, \tag{6}$$

is an enriched φ-contraction with $\varphi(t) = \dfrac{\theta}{b+1} \cdot t$.

2. Any φ-contraction is a $(0, \varphi)$-enriched contraction.

Example 2. *Consider X to be the unit interval $[0,1]$ of $\mathbb{R}$ endowed with the usual norm and the function $T : X \to X$ given by $Tx = 1 - x$, for all $x \in [0,1]$. Then T is neither a contraction nor a φ-contraction but T is an enriched ϕ-contraction (as it is an enriched contraction, see [27]).*

The next fixed point theorem is our first result in this paper and extends many related results in literature, of which we mention Theorem 2.4 in [27], Theorem 5.2 in [5] and Theorem 2.7 in [4].

Theorem 2. *Let $(X, \|\cdot\|)$ be a Banach space and $T : X \to X$ an enriched (b, φ)-contraction. Then*

(i) $Fix\,(T) = \{p\}$;

(ii) There exists $\lambda \in (0,1]$ such that the iterative method $\{x_n\}_{n=0}^\infty$, given by

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n,\ n \ge 0,$$

and $x_0 \in X$ arbitrary, converges strongly to p;

Proof. Choose $x_0 \in X$ and let $\{x_n\}_{n=0}^\infty$, $x_n = T_\lambda x_{n-1} = T_\lambda^n x_0$, $n = 1, 2, \ldots$, be the Picard iteration corresponding to the averaged mapping $T_\lambda(x) := (1-\lambda)x + \lambda T x$, with $\lambda = \dfrac{1}{b+1}$.

By Remark 2.3 in [27] we know that $Fix\,(T) = Fix_\lambda\,(T)$, for any $\lambda \in (0,1]$.

Then, by (5), we obtain (to simplify writing, we use $d(x,y)$ instead of $\|x-y\|$):

$$d(x_n, x_{n+1}) \leq \varphi^n(d(x_0, x_1))$$

which, by (ii$_\varphi$), implies $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$, that is,

$$d(T_\lambda^n x_0, T_\lambda^{n+1} x_0) \to 0, \text{ as } n \to \infty, \tag{7}$$

which expresses the fact that T_λ is asymptotically regular at x_0, for any $x_0 \in X$.

We now prove that $\{x_n\}_{n=0}^\infty$ is a Cauchy sequence. Suppose, on the contrary, that the sequence $\{x_n\}_{n=0}^\infty$ is not Cauchy. Then, there exists $\varepsilon > 0$ and the subsequences $\{x_{n_k}\}_{k=0}^\infty$, $\{x_{m_k}\}_{k=0}^\infty$ of $\{x_n\}_{n=0}^\infty$ with $n_k > m_k > k$ such that

$$d(x_{n_k}, x_{m_k}) \geq \varepsilon, \text{ for all } k \geq 0. \tag{8}$$

We note that, for each k, it is possible to choose a number n_k to be the smallest integer satisfying the above conditions.

Corresponding to the given m_k, it is possible to select n_k to be the smallest integer with $n_k > m_k > k$ and satisfying (8). Then, we have

$$d(x_{n_k-1}, x_{m_k}) < \varepsilon. \tag{9}$$

By (8) and (9), we have

$$\varepsilon \leq d(x_{n_k}, x_{m_k}) \leq d(x_{n_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{m_k}) < \varepsilon + d(x_{n_k}, x_{n_k-1}).$$

that is,

$$\varepsilon \leq d(x_{n_k}, x_{m_k}) < \varepsilon + d(x_{n_k}, x_{n_k-1}). \tag{10}$$

By letting $k \to \infty$ and using (7), we get

$$\lim_{k\to\infty} d(x_{n_k}, x_{m_k}) = \varepsilon. \tag{11}$$

Now, by using once again the triangle inequality, we get

$$d(x_{n_k}, x_{m_k}) \leq d(x_{n_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{m_k-1}) + d(x_{m_k-1}, x_{m_k}) \tag{12}$$

and

$$d(x_{n_k-1}, x_{m_k-1}) \leq d(x_{n_k-1}, x_{n_k}) + d(x_{n_k}, x_{m_k}) + d(x_{m_k}, x_{m_k-1}). \tag{13}$$

Letting $k \to \infty$ in (12) and (13) and using (7) and (11), we have

$$\varepsilon \leq \lim_{k\to\infty} d(x_{n_k-1}, x_{m_k-1})$$

and

$$\lim_{k\to\infty} d(x_{n_k-1}, x_{m_k-1}) \leq \varepsilon,$$

which yields

$$\lim_{k\to\infty} d(x_{n_k-1}, x_{m_k-1}) = \varepsilon.$$

Now, by (5), we have

$$d(x_{n_k}, x_{m_k}) = d(T_\lambda x_{n_k-1}, T_\lambda x_{m_k-1}) \leq \varphi(d(x_{n_k-1}, x_{m_k-1})), \text{ for all } k \geq 0.$$

We let $k \to \infty$ in the above inequality and use the continuity of T_λ to get

$$\varepsilon \leq \varphi(\varepsilon),$$

which contradicts (iii_φ), as $\varepsilon > 0$.

Thus, $\{T_\lambda^n x_0\}_{n\in\mathbb{N}}$ is Cauchy and therefore $\{T_\lambda^n x_0\}_{n\in\mathbb{N}}$ is convergent. Let $p = \lim_{n\to\infty} T_\lambda^n x_0$. Hence,

$$p = T_\lambda \left(\lim_{n\to\infty} T_\lambda^{n-1} x_{n-1} \right) = T_\lambda p,$$

which shows that $p \in Fix\,(T_\lambda)$.

By assuming that there would exist $q \in Fix\,(T_\lambda)$, such that $q \neq p$, then it follows $d(p,q) > 0$ and thus, by the φ-contractiveness condition (5) we are lead to

$$0 < d(p,q) = d(T_\lambda p, T_\lambda q) \leq \varphi(d(p,q)) < d(p,q),$$

a contradiction. □

The fact that Theorem 2 is an effective generalization of Theorem 2.4 in [27], Theorem 5.2 in [5], Theorem 2.7 in [4] etc. follows by means of the next example.

Example 3. *Let $I = \left[0, \frac{\pi}{4}\right]$ and $\mathcal{C}(I)$ the Banach space of continuous real functions defined on I with the supremum norm $\|\cdot\|_\infty$.*

Let $T\colon \mathcal{C}(I) \to \mathcal{C}(I)$ be the operator defined by

$$Tx(t) = \frac{4}{\pi} \arctan\left(\int_0^t |x(s)| \mathrm{d}s \right).$$

In the sequel, we prove that T is a enriched φ-contraction. In fact, for $t \in [0, \frac{\pi}{4}]$, by taking into account that $|\arctan x - \arctan y| \leq \arctan(|x-y|)$, we have

$$\begin{aligned}
|(Tx)(t) - (Ty)(t)| &= \frac{4}{\pi} \left| \arctan\left(\int_0^t |x(s)| \mathrm{d}s \right) - \arctan\left(\int_0^t |y(s)| \mathrm{d}s \right) \right| \\
&\leq \frac{4}{\pi} \arctan\left(\left| \int_0^t |x(s)| - \int_0^t |y(s)| \mathrm{d}s \right| \right) \\
&= \frac{4}{\pi} \arctan\left(\left| \int_0^t (|x(s)| - |y(s)|) \mathrm{d}s \right| \right) \\
&\leq \frac{4}{\pi} \arctan\left(\int_0^t ||x(s)| - |y(s)|| \mathrm{d}s \right) \\
&\leq \frac{4}{\pi} \arctan\left(\int_0^t |x(s) - y(s)| \mathrm{d}s \right) \\
&\leq \frac{4}{\pi} \arctan\left(\|x-y\|_\infty \int_0^t \mathrm{d}s \right) \\
&= \frac{4}{\pi} \arctan(\|x-y\|_\infty t) \\
&\leq \frac{4}{\pi} \arctan\left(\frac{\pi}{4} \|x-y\|_\infty \right)
\end{aligned}$$

Therefore,

$$\begin{aligned}
d(Tx, Ty) &= \sup_{t\in[0,\frac{\pi}{4}]} |(Tx)(t) - (Ty)(t)| \\
&\leq \frac{4}{\pi} \arctan\left(\frac{\pi}{4} d(x,y) \right).
\end{aligned}$$

If we consider as $\varphi(t) = \frac{4}{\pi} \arctan\left(\frac{\pi}{4} t\right)$ it is clear that φ is a comparison function and, consequently, $Tx = T_\lambda x$ (with $\lambda = 1$) is an enriched φ-contraction.

In the sequel, we will prove that T is not a (b,θ)-enriched contraction.

We take $x(t) = 1$ and $y(t) = 0$ as functions in $\mathcal{C}(I)$ and $b \geq 0$, then

$$\begin{aligned}\|b(x-y)+Tx-Ty\| &= \sup_{t\in[0,\frac{\pi}{4}]}\left|b+\frac{4}{\pi}\arctan\left(\int_0^t 1\mathrm{d}s\right)\right| \\ &= \sup_{t\in[0,\frac{\pi}{4}]}\left|b+\frac{4}{\pi}\arctan t\right| \\ &= b+\frac{4}{\pi}\cdot\frac{\pi}{4} \\ &= b+1 \\ &= (b+1)\|x-y\|.\end{aligned}$$

As θ must be in the interval $[0, b+1)$, the above result shows that (6) cannot be satisfied. This proves that, indeed, T is not a (b,θ)-enriched contraction.

Theorem 2 cannot be applied in the case when T is not an (enriched) φ-contraction in the sense of our Definition 1 but only a certain iterate of it is an (enriched) φ-contraction (see Example 2 in [27]). In such kind of situations the next result is useful.

Corollary 1. *Let $(X, \|\cdot\|)$ be a Banach space and let $U : X \to X$ be a mapping such that for a positive integer N, the iterate U^N is an enriched φ-contraction. Then,*

(i) Fix $(U) = \{p\}$, for some $p \in X$.

(ii) One can find a constant $\mu \in [0,1)$ such that the sequence $\{x_n\}_{n=0}^{\infty}$ given by

$$x_{n+1} = (1-\mu)x_n + \mu U^N x_n \; n \geq 0,$$

converges strongly to p, for any $x_0 \in X$.

Proof. The proof follows by applying Theorem 2 (i) in the case $T = U^N$ and thus one obtains that *Fix* $(U^N) = \{p\}$. It is easy to see that

$$U^N(U(p)) = U^{N+1}(p) = U(U^N(p)) = U(p),$$

which shows that $U(p)$ is a fixed point of U^N.

Since U^N has a unique fixed point, p, it follows that $U(p) = p$, i.e., $p \in$ *Fix* (U).

The remaining conclusion (ii) follows by Theorem 2, too. □

3. Geraghty Type Fixed Point Theorems for Enriched ψ-Contractions

According to [20], one considers the auxiliary functions $\psi : \mathbb{R}_+ \to [0,1)$ satisfying the following property:

(g) If $\{t_n\} \subset \mathbb{R}_+$ and $\psi(t_n) \to 1$ as $n \to \infty$, then $t_n \to 0$ as $n \to \infty$.

Let $\mathcal{P}$ denote the set of all auxiliary functions ψ satisfying condition (g) above. It is easy to check that $\mathcal{P} \neq \emptyset$, as the function $\psi(t) = \exp(-t)$, for $t \geq 0$, belongs to $\mathcal{P}$.

The main result of this section is the following Geraghty type fixed point theorems for enriched ψ-contractions.

Theorem 3. *Let $(X, \|\cdot\|)$ be a Banach space and let $T : X \to X$ be an enriched ψ-contraction, i.e., a mapping for which there exists a function $\psi \in \mathcal{P}$ such that*

$$\|b(x-y)+Tx-Ty\| \leq (b+1)\psi(\|x-y\|)\|x-y\|, \forall x,y \in X. \tag{14}$$

Then,

(i) Fix $(T) = \{p\}$, for some $p \in X$.

(*ii*) *The sequence* $\{x_n\}_{n=0}^{\infty}$ *obtained from the iterative process*

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n,\ n \geq 0, \tag{15}$$

and $x_0 \in X$ *arbitrary, converges strongly to* p.

Proof. By condition (14), if we denote $\lambda = \dfrac{1}{b+1}$, we infer that T_λ satisfies the following condition

$$\|T_\lambda x - T_\lambda y\| \leq \psi(\|x-y\|)\|x-y\|, \forall x, y \in X, \tag{16}$$

where $T_\lambda(x) := (1-\lambda)x + \lambda Tx, \lambda \in (0,1]$, is the averaged mapping associated to T. To simplify writing, we work with $d(x,y)$ instead of $\|x-y\|$.

Choose $x_0 \in X$ and construct the Picard iteration associated to T_λ, i.e., the sequence $\{x_n\}_{n=0}^{\infty}$ given by

$$x_n = T_\lambda x_{n-1} = T_\lambda^n x_0,\ \ n = 1, 2, \dots,$$

Assume there exists $n \geq 0$ such that $x_{n+1} = x_n$. In this case $Fix\,(T_\lambda) = \{x_n\} = Fix\,(T)$ and the proof is finished.

Otherwise, assume that $x_{n+1} \neq x_n$, for all $n \geq 0$. Then, by the contraction condition (16), one obtains

$$d(x_{n+1}, x_{n+2}) \leq \psi(d(x_n, x_{n+1}))d(x_n, x_{n+1}) \tag{17}$$

which implies that the sequence of nonnegative real numbers $\{d(x_n, x_{n+1})\}$ is decreasing, hence convergent. Denote

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = r \geq 0.$$

Suppose first that $r > 0$. In this case, it follows that all terms of the sequence $\{d(x_n, x_{n+1})\}$ are positive and, thus, by (17), we get

$$\frac{d(x_{n+1}, x_{n+2})}{d(x_n, x_{n+1})} \leq \psi(d(x_n, x_{n+1})) < 1.$$

We now let $n \to \infty$ in the previous inequalities to get

$$\lim_{n\to\infty} \psi(d(x_n, x_{n+1})) = 1.$$

Since $\psi \in \mathcal{P}$, this implies

$$d(x_{n+1}, x_{n+2}) = d(T_\lambda^n x_0, T_\lambda^{n+1} x_0) \to 0, \text{ as } n \to \infty. \tag{18}$$

Inequality (18) expresses the fact that the mapping T_λ is asymptotically regular at x_0, for any $x_0 \in X$.

The same conclusion follows in the case $r = 0$, when we have

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

To prove that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence we proceed by contradiction. So, suppose that $\{x_n\}_{n=0}^{\infty}$ is not Cauchy.

Then, there exists $\varepsilon > 0$ and two subsequences $\{x_{n_k}\}_{k=0}^{\infty}$, $\{x_{m_k}\}_{k=0}^{\infty}$ of $\{x_n\}_{n=0}^{\infty}$ such that $n_k > m_k > k$ and

$$d(x_{n_k}, x_{m_k}) \geq \varepsilon. \tag{19}$$

For the above given m_k, we can choose n_k to be the smallest integer with $n_k > m_k > k$ that satisfies (19). Then, we have

$$d(x_{n_k - 1}, x_{m_k}) < \varepsilon \tag{20}$$

and by (19) and (20), one obtains

$$\varepsilon \le d(x_{n_k}, x_{m_k}) \le d(x_{n_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{m_k}) < \varepsilon + d(x_{n_k}, x_{n_k-1}),$$

that is,

$$\varepsilon \le d(x_{n_k}, x_{m_k}) < \varepsilon + d(x_{n_k}, x_{n_k-1}). \tag{21}$$

We let $k \to \infty$ and use (18) to obtain

$$\lim_{k\to\infty} d(x_{n_k}, x_{m_k}) = \varepsilon. \tag{22}$$

Now, by using the triangle inequality, one obtains

$$d(x_{n_k}, x_{m_k}) \le d(x_{n_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{m_k-1}) + d(x_{m_k-1}, x_{m_k}) \tag{23}$$

and

$$d(x_{n_k-1}, x_{m_k-1}) \le d(x_{n_k-1}, x_{n_k}) + d(x_{n_k}, x_{m_k}) + d(x_{m_k}, x_{m_k-1}). \tag{24}$$

By letting $k \to \infty$ in (23) and (24) and using (18) and (22), we have

$$\varepsilon \le \lim_{k\to\infty} d(x_{n_k-1}, x_{m_k-1})$$

and

$$\lim_{k\to\infty} d(x_{n_k-1}, x_{m_k-1}) \le \varepsilon,$$

which yields

$$\lim_{k\to\infty} d(x_{n_k-1}, x_{m_k-1}) = \varepsilon.$$

Next, by (17), we have

$$d(x_{n_k}, x_{m_k}) \le \psi(d(x_{n_k-1}, x_{m_k-1}))d(x_{n_k-1}, x_{m_k-1}) < d(x_{n_k-1}, x_{m_k-1})$$

and, by letting $k \to \infty$ in the previous inequality, we get

$$\varepsilon \le \lim_{k\to\infty} \psi(d(x_{n_k-1}, x_{m_k-1}) \cdot \varepsilon \le \varepsilon,$$

which implies

$$\lim_{k\to\infty} \psi(d(x_{n_k-1}, x_{m_k-1}) = 1.$$

Since $\psi \in \mathcal{P}$, we obtain

$$\lim_{k\to\infty} d(x_{n_k-1}, x_{m_k-1}) = 0.$$

This result and (22) yields $\varepsilon = 0$, which is a contradiction.

Thus, $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence and, since X is a Banach space, $\{x_n\}_{n=0}^{\infty}$ is convergent. Denote

$$\lim_{n\to\infty} x_n = p. \tag{25}$$

Then,

$$d(p, T_\lambda p) \le d(p, T_\lambda x_n) + d(T_\lambda x_n, T_\lambda p) \le$$
$$d(p, x_{n+1}) + \psi(d(x_n, p)) \cdot d(x_n, p).$$

Letting $n \to \infty$ in the previous inequality, we get $d(p, T_\lambda p) = 0$, that is, p is a fixed point of T_λ.

To prove the uniqueness, we suppose that there exists $q \in Fix(T_\lambda)$, $q \ne p$. Then, $d(p, q) > 0$ and, by (14), we have

$$d(p, q) = d(T_\lambda p, T_\lambda q) \le \psi(d(p, q))d(p, q) < d(p, q),$$

a contradiction.
□

Remark 2. *Theorem 3 is a very general result: a particular case of it is Theorem 2 in this paper and in the following we also enumerate some other important particular cases of it.*

(1) The Geraghty fixed point theorem (see [20,29]) is obtained from Theorem 3 by taking $\lambda = 1$.

(2) From Theorem 3 we also obtain the pioneering fixed point result of Rakotch ([19], p. 463) in the particular case $\lambda = 1$ *and* $\psi(t)$ *not increasing.*

(3) Browder's fixed point theorem ([14], p. 27) can be obtained from Theorem 3 for $\lambda = 1$ *and* $\psi(t) = \frac{\varphi(t)}{t}$, *where* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *is right continuous, nondecreasing and satisfies* $\varphi(t) < t$ *for* $t > 0$.

(4) By Theorem 3 we also obtain Boyd and Wong's fixed point theorem ([15], p. 331) if we take $\lambda = 1$ *and* $\psi(t) = \frac{\varphi(t)}{t}$, *where* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *is right upper continuous, nondecreasing and such that* $\varphi(t) < t$ *for* $t > 0$.

(5) We obtain Matkowski's fixed point theorem ([17]) (see also [30]) from Theorem 3 in the particular case $\lambda = 1$ *and* $\psi(t)$ *of the form* $\frac{\varphi(t)}{t}$, *with* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *a comparison function.*

4. Cyclic Enriched φ-Contractions

The aim of this section is to extend further the class of enriched φ-contractions by means of the concept of cyclical mapping. This direction of extending the Banach contraction mapping has been open by Kirk, Srinivasan and Veeramany [31] in 2003, who considered mappings $T : A \cup B \to A \cup B$ satisfying a cyclic condition of the form:

$$d(Tx, Ty) \le a d(x, y), \forall x \in A \text{ and } \forall y \in B, \tag{26}$$

where A, B are nonempty closed subsets of a metric space X such that

$$T(A) \subset B \text{ and } T(B) \subset A, \tag{27}$$

and $a \in (0, 1)$ is a constant.

We note that, if $A = B = X$, then the cyclical condition (26) reduces to the Banach contraction condition (1).

The first main result (Theorem 1.1 in [31]) is an interesting generalization of the Banach contraction mapping principle to cyclical mappings.

Theorem 4. *Let* A *and* B *be two non-empty closed subsets of a complete metric space* X, *and suppose* $T : X \to X$ *satisfies (26) and (27) above. Then* T *has a unique fixed point in* $A \cap B$.

A more general context for the study of cyclic phenomena in connection with the fixed point problem has been introduced by Rus [32].

Let X be a nonempty set, m a positive integer and $T : X \to X$ an operator. By definition, $\bigcup_{i=1}^m X_i$ is a *cyclic representation of* X *with respect to* T if

(i) $X_i \neq \emptyset$, $i = 1, 2, ..., m$;

(ii) $T(X_1) \subset X_2, \ldots, T(X_{m-1}) \subset X_m$, $T(X_m) \subset X_1$.

Based on this notion, Păcurar and Rus [33] introduced the concept of cyclic φ-contraction as follows.

Let (X, d) be a metric space, m a positive integer, $A_1, \ldots, A_m$ nonempty and closed subsets of X and $Y = \bigcup_{i=1}^m A_i$. An operator $T : X \to X$ is called a *cyclic φ-contraction* if

(a) $\bigcup_{i=1}^m A_i$ is a cyclic representation of Y with respect to T;

(b) there exists a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$d(Tx, Ty) \le \varphi(d(x, y)), \tag{28}$$

for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, \ldots, m$, where $A_{m+1} = A_1$.

We note that if $m = 2$, $A_1 := A$, $A_2 := B$ and $\varphi(t) = at$, then the cyclic condition (28) reduces to (26) (and therefore to (1) when $A = B = X$).

The mai result in [33] is an existence, uniqueness and approximation result of the fixed points of cyclic φ-contractions in a metric space, in the case φ is a comparison function possessing some appropriate properties (i.e., φ is a (c)-comparison function, see [34]).

Having as staring point the results in Section 1 of this paper, on the one hand, and the above mentioned results from [33], on the other hand, our aim in this section is to introduce and study the class of enriched cyclic φ-contractions in the setting of a real Banach space.

Definition 2. *Consider a linear normed space $(X, \|\cdot\|)$, $T : X \to X$ be a self mapping and let $\bigcup_{i=1}^{m} A_i$ be a cyclic representation of X with respect to T. If one can find a constant $b \in [0, +\infty)$ and a comparison function φ such that*

$$\|b(x-y) + Tx - Ty\| \leq (b+1)\varphi(\|x-y\|), \forall x \in A_i \text{ and } \forall y \in A_{i+1}, \tag{29}$$

for $i = 1, 2, \ldots, m$, where $A_{m+1} = A_1$, then T is said to be a cyclic enriched φ-contraction.

Example 4. *(1) Any cyclic φ-contraction is a cyclic enriched φ-contraction (with $b = 0$);*

(2) Any enriched contraction [27] is a cyclic enriched φ-contraction (with $m = 1$).

A comparison function φ is said to be a (c)-comparison function (see [34]) if there exist $k_0 \in \mathbb{N}$, $\delta \in (0,1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that

$$\varphi^{k+1}(t) \leq \delta\varphi^k(t) + v_k,\ k \geq k_0,\ t \in \mathbb{R}_+. \tag{30}$$

It is known (see for example Lemma 1.1 in [33]) that if $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a (c)-comparison function, then $s : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$s(t) = \sum_{k=1}^{\infty} \varphi^k(t),\ t \in \mathbb{R}_+, \tag{31}$$

is increasing and continuous at 0.

Now we are ready to state the main result of this section.

Theorem 5. *Let $(X, \|\cdot\|)$ be a Banach space, m a positive integer, $A_1, \ldots, A_m$ nonempty and closed subsets of X, $Y = \bigcup_{i=1}^{m} A_i$ and $T : X \to X$ a cyclic enriched φ-contraction with φ a (c)-comparison function. Then*

(i) T has a unique fixed point $p \in \bigcap_{i=1}^{m} A_i$;

(ii) there exists $\lambda \in (0,1]$ such that the iterative method $\{x_n\}_{n=0}^{\infty}$, given by

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n,\ n \geq 0,$$

and $x_0 \in X$ arbitrary, converges strongly to p;

(iii) the following estimates hold

$$\|x_n - p\| \leq s(\varphi^n(\|x_0 - x_1\|)), n \geq 1;$$

$$\|x_n - p\| \leq s(\varphi(\|x_n - x_{n+1}\|)), n \geq 1;$$

(iv) for any $x \in Y$:

$$\|x_n - p\| \leq s(\lambda\|x - Tx\|),$$

where s is defined by (31).

Proof. Choose $x_0 \in Y = \bigcap_{i=1}^{m} A_i$ and define the iterative scheme $\{x_n\}$ by

$$x_n = T_\lambda x_{n-1},\ n \geq 1, \tag{32}$$

where $T_\lambda(x) := (1-\lambda)x + \lambda Tx$, with $\lambda = \dfrac{1}{b+1}$.

Then, by (29), we obtain:

$$\|T_\lambda x - T_\lambda y\| \leq \varphi(\|x-y\|),\ \forall x \in A_i \text{ and } \forall y \in A_{i+1}, \tag{33}$$

for all $i = 1, 2, \ldots, m$, which shows that T_λ is a cyclic φ-contraction.

Now, by applying Theorem 2.1 in [33], we obtain that T_λ has a unique fixed point $p \in \bigcap_{i=1}^{m} A_i$ and since T and T_λ share the same set of fixed points, this proves (i) and (ii).

To prove (iii) and (iv) we essentially use the properties of the (c)-comparison function φ and that of s. □

Remark 3. *(1) In the particular case $b = 0$, by Theorem 5 we obtain Theorem 2.1 in [33] which, in turn, generalizes many important results from fixed point theory;*

(2) We note that the error estimates in Theorem 5 are important from a practical point of view in the approximation of solutions of functional equations;

(3) By using the concepts from Section 3 in [33], one can establish various interesting results for the class of cyclic enriched φ-contractions, like: well posedness of the fixed point problem, the limit shadowing property, data dependence of the fixed points etc.

We end this section by stating a Maia type fixed point result that extends further Theorem 5. Its proof is similar to the ones in [35] and is left as exercise for the reader.

Theorem 6. *Let X be a linear vector space endowed with a metric d and a norm $\|\cdot\|$ satisfying the condition*

$$d(x,y) \leq \|x-y\|, \text{ for all } x, y \in X. \tag{34}$$

Let m be a positive integer, $A_1, \ldots, A_m$ nonempty and closed subsets of X, $Y = \bigcup_{i=1}^{m} A_i$ and $T : Y \to Y$.

Suppose

(i) $\bigcup_{i=1}^{m} A_i$ is a cyclic representation of Y with respect to T;

(ii) T a cyclic enriched φ-contraction with respect to $\|\cdot\|$, with φ a (c)-comparison function;

(iii) (Y, d) is a complete metric space;

(iv) $T : (Y, d) \to (Y, d)$ is continuous.

Then

(i) Fix $(T) = \{p\}$;

(ii) There exists $\lambda \in (0, 1]$ such that the iterative method $\{x_n\}_{n=0}^{\infty}$, given by

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n,\ n \geq 0, \tag{35}$$

converges in (X, d) to p, for any $x_0 \in X$.

Remark 4. *In the particular case $d(x,y) = \|x-y\|$, by Theorem 6 we obtain Theorem 5, while, for $b = 0$, we obtain Theorem 3.7 in [33].*

5. An Application to Second Order (p,q)-Difference Equations with Integral Boundary Value Conditions

As an illustration of the usefulness of the fixed point results established in the previous sections, we present an existence and uniqueness result for a second order (p,q)-difference equation with integral boundary value conditions of the form

$$\begin{cases} D^2_{p,q}u(t) + f(t,u(t)) = 0, \ t \in (0,1), \\ u(0) = \int\limits_0^1 u(t)d_{p,q}t, \\ u(1) = \int\limits_0^1 tu(t)d_{p,q}t, \end{cases} \tag{36}$$

where p,q are such that $0 < q < p \le 1$, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a given function and $D_{p,q}$ is the (p,q)-difference operator, defined as follows (for more details, see for example [36]).

Assume $u : [0,T] \to \mathbb{R}$, $T > 0$, is a given function and p,q are such that $0 < q < p \le 1$. We also assume, without any loss of generality, that $p + q \ne 1$.

The (p,q)-*derivative* of u, denoted by $D_{p,q}u(t)$, is defined (see for example [26]), by

$$D_{p,q}u(t) = \frac{u(pt) - u(qt)}{(p-q)t}, \text{ if } t \ne 0 \tag{37}$$

and

$$D_{p,q}u(0) = \lim_{t\to 0} D_{p,q}u(t), \text{ if } t = 0. \tag{38}$$

One can see that $D_{p,q}u(t)$ is defined on the larger interval $[0,T/p]$, which includes the interval $[0,T]$ on which u is defined.

We say that the function u is (p,q)-*differentiable* if $D_{p,q}u(t)$ exists for all $t \in [0,T/p]$.

The (p,q)-*integral* of u, denoted by $\int\limits_0^t u(s)d_{p,q}s$ is by definition

$$\int\limits_0^t u(s)d_{p,q}s = \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} u\left(\frac{q^n}{p^{n+1}}t\right), \tag{39}$$

whenever the series in the right hand side of (39) converges.

In contrast to the case of (p,q)-derivative of u, the (p,q)-integral of u is defined on the interval $[0,pT]$ which is strictly included in the interval $[0,T]$.

The (p,q)-differentiation and (p,q)-integration have similar properties to the ones in the case of q-calculus and usual calculus, see for example Theorems 2.3 and 2.4 and Lemma 2.8 in [26]. For the sake of completeness, we state in the following

Lemma 1 (Lemma 2.8, [26])**.** *For any $h \in C([0,1],\mathbb{R})$, the boundary value problem*

$$\begin{cases} D^2_{p,q}u(t) + h(t) = 0, \ t \in (0,1), \\ u(0) = \int\limits_0^1 u(t)d_{p,q}t, \ u(1) = \int\limits_0^1 tu(t)d_{p,q}t \end{cases} \tag{40}$$

is equivalent to the integral equation

$$u(t) = -\frac{1}{p}\int_0^t (t-qs)h\left(\frac{s}{p}\right)d_{p,q}s + \frac{1}{p}\cdot\frac{p+q}{p+q-1}\int_0^1 (1-qs)h\left(\frac{s}{p}\right)d_{p,q}s$$
$$-\frac{p^2-q^2}{p^3(p+q-1)(p^2+pq+q^2)}\int_0^1 (s-qs^2)$$

$$\left[(p^2+pq+q^2)(1-t)(p+q-1)+s+p^2+(p+q)(q-1)\right]h\left(\frac{s}{p^2}\right)d_{p,q}s. \quad (41)$$

We can now state our main result in this section. To this end, we denote for brevity

$$\delta = \frac{1}{p} + \frac{p-q}{p(p^2+pq+q^2)} + \frac{1}{(p+q-1)(p^2+pq+q^2)} + \frac{(p-q)(p^2+(p+q)(q-1))}{p(p+q-1)(p^2+pq+q^2)}.$$

Theorem 7. *Suppose that $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a continuous function for which there exists a comparison function $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ such that:*

$$|f(t,u)-f(t,v)| \le \varphi(|u-v|), \forall t\in[0,1] \text{ and } u,v\in\mathbb{R}. \quad (42)$$

If $\delta L<1$, where $L=\sup\limits_{t\in\mathbb{R}_+}\frac{\varphi(t)}{t}$, then the boundary value problem (36) has a unique solution $u^*\in C([0,1],\mathbb{R})$.

Proof. In view of Lemma 1, the boundary value problem (36) is equivalent to the fixed point problem

$$x = Tx, \quad (43)$$

where $T:X\to X$ is the integral operator defined by the right hand side of (41) and $X=C([0,1],\mathbb{R})$.

It is well known that X is a Banach space with respect to the sup norm, defined by

$$\|u\| = \sup_{t\in[0,1]}|u(t)|, \forall u\in X.$$

Denote $M=\sup\limits_{t\in[0,1]}|f(t,0)|$, choose a constant R such that

$$R \ge \frac{M\delta}{1-L\delta}$$

and consider the closed ball $B_R=\{u\in X:\|u\|\le R\}$.

First, we prove that B_R is invariant with respect to T, that is, $T(B_R)\subset B_R$.

For $u\in B_R$, we obtain after straightforward calculations (see the proof of Theorem 3.1 in [26], for details) that

$$\|Tu\| \le (LR+M)\delta \le R,$$

which proves that $T(B_R)\subset B_R$.

Next, we prove that T is a φ-contraction. For any $u,v\in B_R$ and $t\in[0,1]$, we have

$$\sup_{t\in[0,1]}|(Tu)(t)-(Tv)(t)| =$$

$$\le \frac{1}{p}\sup_{t\in[0,1]}\int_0^t|t-qs|\cdot|f(s,u(ps))-f(s,v(ps))|d_{p,q}s$$

$$+\frac{1}{p}\cdot\frac{p+q}{p+q-1}\sup_{t\in[0,1]}\int_0^1|1-qs|\cdot|f(s,u(ps))-f(s,v(ps))|d_{p,q}s$$

$$+\frac{p^2-q^2}{p^3(p+q-1)(p^2+pq+q^2)}.$$

$$\sup_{t\in[0,1]} \int_0^1 B(s)\Big|f(s,u(p^2s)) - f(s,v(p^2s))\Big|d_{p,q}s,$$

where

$$B(s) = |s - qs^2|\cdot$$

$$\Big[(p^2+pq+q^2)(1-t)(p+q-1) + s + p^2 + (p+q)(q-1)\Big].$$

By using the Lipschitz type inequality (42) and by taking the sup in both sides of the obtained inequality above, one obtains

$$\|Tu - Tv\| \le \delta\varphi(\|u-v\|), \forall u,v \in B_R.$$

Using the hypotheses, we deduce that $\psi(t) = \delta\varphi(t)$ is a comparison function.

This proves that $T : B_R \to B_R$ is a ψ-contraction and since B_R is closed, the conclusion follows by applying Theorem 2. □

Remark 5. *(1) If in Theorem 7 we have $\varphi(t) = Lt$, then one obtains the main result (Theorem 3.1 in [26]);*

(2) If the comparison function in Theorem 7 is in particular a (c)-comparison function, then it is possible to also obtain results on the approximation of the solution of the second order (p,q)-difference equation with integral boundary value conditions (36).

6. Conclusions

1. We introduced the class of enriched φ-contractions in Banach spaces as a natural generalization of φ-contractions and then studied the existence and approximation of the fixed points for mappings in this new class.

2. According to the terminology introduced in [28], we proved that the class of enriched φ-contractions is an unsaturated class of mappings in the setting of a Banach space, which means that the concept of enriched φ-contraction is an effective generalization of that of φ-contraction. This fact is illustrated by appropriate examples (Examples 1, 2 and 3).

3. The obtained results are very general and include as particular cases, in the setting of a Banach space, many previous results in literature, see Remark 2.

4. Some related results were obtained in convex metric spaces [37], but in this setting it was not possible to prove that enriched φ-contractions form a unsaturated class of mappings. It remains an open problem wether in other convex metric spaces which are not Banach spaces, the class of enriched φ-contractions constitutes a saturated or an unsaturated class of contractive type mappings.

5. In Section 4, we extended the results in Section 2 to the class of enriched cyclic φ-contraction and we also established a Maia type fixed point theorem.

6. In the last section of the paper we illustrated the usefulness of our fixed point results by studying the existence and uniqueness of the solutions of some second order (p,q)-difference equations with integral boundary value conditions.

7. Similar developments could be considered, starting from the new interesting results in Gornicki and Bisht [38] and in some other related papers like [35,39–50] etc.

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