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Iterated Partial Sums of the k-Fibonacci Sequences

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Abstract: In this paper, we find the sequence of partial sums of the k-Fibonacci sequence, say, $S_{k,n} = \sum_{j=1}^{n} F_{k,j}$, and then we find the sequence of partial sums of this new sequence, $S_{k,n}^{2)} = \sum_{j=1}^{n} S_{k,j}$, and so on. The iterated partial sums of k-Fibonacci numbers are given as a function of k-Fibonacci numbers, in powers of k, and in a recursive way. We finish the topic by indicating a formula to find the first terms of these sequences from the k-Fibonacci numbers themselves.

Keywords: partial sums; k-Fibonacci sequences

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1. Introduction

Recently, the iterated partial sums of the classical Fibonacci sequence have been studied [1]. Surprisingly, as noted in [1], these iterated partial sums are related to Schreier sets, which were used to solve a problem in Banach space theory [2]. In combinatorics, these numbers are related to Ramsey-type theorems for subsets of $\mathbb{N}$. Our purpose in this paper is to extend this study to the k-Fibonacci sequences and expand its results. Analogous identities may be obtain for other sequences as the k-Lucas numbers [3].

The k-Fibonacci numbers appear when studying the four-triangle longest-edge (4T-LE) partition of triangles, as another example of the relation between geometry and numbers [4]. The 4T-LE partition of a triangle is obtained by joining the middle point of the longest edge of the triangle to the opposite vertex and to the midpoints of the two remaining edges. This partition and the associated refinement algorithm were introduced by M.-C. Rivara [5], and their extensions to higher dimensions have been used in finite element methods [6].

The k-Fibonacci numbers are defined by the recurrence relation $F_{k,n} = k\,F_{k,n-1} + F_{k,n-2}$ with initial conditions $F_{k,0} = 0$ and $F_{k,1} = 1$. The first k-Fibonacci numbers are $1, k, k^2+1, k^3+2k, k^4+3k^2+1, k^5+4k^3+3k, k^6+5k^4+6k^2+1, \ldots$

The associated characteristic equation is $r^2 - k\,r - 1 = 0$, and its solutions are $\sigma_1 = \frac{k+\sqrt{k^2+4}}{2}$ and $\sigma_2 = \frac{k-\sqrt{k^2+4}}{2}$. These roots verify that $\sigma_1 \cdot \sigma_2 = -1$, $\sigma_1 + \sigma_2 = k$.

In [4], the following formulas, among others, are proven:

Generating function:

$$\sum_{n=0}^{\infty} F_{k,n} x^n = \frac{x}{1-kx-x^2}. \tag{1}$$

Binet formula:

$$F_{k,n} = \frac{\sigma_1^n - \sigma_2^n}{\sigma_1 - \sigma_2}. \tag{2}$$

Sum of the first terms:

$$\sum_{j=0}^{n} F_{k,j} = \frac{1}{k}(F_{k,n+1} + F_{k,n} - 1). \tag{3}$$

2. Iterated Partial Sums

The definition of partial sums of the k-Fibonacci sequence is introduced, along with how to find them and some of the relationships between their elements.

Definition 1. *For $r \geq 1$, the iterated partial sums of the k-Fibonacci numbers are defined as $S_{k,n}^{r)} = \sum_{j=1}^{n} S_{k,j}^{r-1)}$, with initial condition $S_{k,n}^{0)} = F_{k,n}$.*

The Table 1 shows the first elements of these sequences.

Table 1. Iterated partial sums of the k-Fibonacci sequences.

r \ n	**1**	**2**	**3**	**4**
0	$F_{k,1}$	$F_{k,2}$	$F_{k,3}$	$F_{k,4}$
1	$F_{k,1}$	$F_{k,2}+F_{k,1}$	$F_{k,3}+F_{k,2}+F_{k,1}$	$F_{k,4}+F_{k,3}+F_{k,2}+F_{k,1}$
2	$F_{k,1}$	$F_{k,2}+2F_{k,1}$	$F_{k,3}+2F_{k,2}+3F_{k,1}$	$F_{k,4}+2F_{k,3}+3F_{k,2}+4F_{k,1}$
3	$F_{k,1}$	$F_{k,2}+3F_{k,1}$	$F_{k,3}+3F_{k,2}+6F_{k,1}$	$F_{k,4}+3F_{k,3}+6F_{k,2}+10F_{k,1}$
4	$F_{k,1}$	$F_{k,2}+4F_{k,1}$	$F_{k,3}+4F_{k,2}+10F_{k,1}$	$F_{k,4}+4F_{k,3}+10F_{k,2}+20F_{k,1}$

2.1. First Formula

The following formula allows us to find any term of these sequences in a non-recursive way.

Theorem 1. *For $r \geq 1$:*

$$S_{k,n}^{r)} = \sum_{j=0}^{n} \binom{r+j-1}{j} F_{k,n-j}. \tag{4}$$

Proof. Notice that the right-side hand is the convolution of sequences $\left\{\binom{r+n-1}{n}\right\}_{n\geq 0}$, and $\{F_{k,n}\}_{n\geq 0}$. Since their respective generating functions [4,7] are $\frac{1}{(1-x)^r}$ and $\frac{x}{1-kx-x^2}$, the conclusion follows. □

For instance, $S_{k,4}^{3)} = \sum_{j=0}^{4} \binom{2+j}{j} F_{k,4-j} = F_{k,4} + 3F_{k,3} + 6F_{k,2} + 10F_{k,1}$.

Moreover, the first n addends of $S_{k,n+1}^{r)}$ are the same as those of $S_{k,n}^{r)}$ without more than changing $F_{k,n}$ by $F_{k,n+1}$. The last addend of $S_{k,n+1}^{r)}$ is $\binom{n+r-2}{n-1}$, because $F_{k,1} = 1$.

If $r = 1$: $S_{k,n}^{1)} = \sum_{j=0}^{n} \binom{j}{j} F_{k,n-j} = \sum_{j=0}^{n} F_{k,j}$.

Remark 1. *Notice that for any sequence $\{a_{k,n}\}_{n\geq 0}$, if $S_{k,n}^{r)}$ denotes the r-th iterated partial sum of $\{a_{k,n}\}_{n\geq 0}$, then as in the previous theorem, $S_{k,n}^{r)} = \sum_{j=0}^{n} \binom{r+j-1}{j} a_{k,n-j}$.*

2.2. Partial Sums in Powers of k

By applying the definition of the k-Fibonacci numbers in Table 1, for $r = 0, 1, 2, 3, 4$, the following sequences are obtained in Tables 2–5:

Table 2. Iterated partial sums of the k-Fibonacci sequences.

r \ n	**1**	**2**	**3**	**4**	**5**
0	1	k	k^2+1	k^3+2k	k^4+3k^2+1
1	1	$k+1$	k^2+k+2	k^3+k^2+3k+2	$k^4+k^3+4k^2+3k+3$
2	1	$k+2$	k^2+2k+4	k^3+2k^2+5k+6	$k^4+2k^3+6k^2+8k+9$
3	1	$k+3$	k^2+3k+7	$k^3+3k^2+8k+13$	$k^4+3k^3+9k^2+16k+22$
4	1	$k+4$	$k^2+4k+11$	$k^3+4k^2+12k+24$	$k^4+4k^3+13k^2+28k+46$

For $k=1$ in Table 2, the respective sequences are

Table 3. Iterated partial sums of the classical Fibonacci sequence.

r \ n	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
0	1	1	2	3	5	8	13	21	34	55	89	144
1	1	2	4	7	12	20	33	54	88	143	232	376
2	1	3	7	14	26	46	79	133	221	364	596	972
3	1	4	11	25	51	97	176	309	530	894	1490	2462
4	1	5	16	41	92	189	365	674	1204	2098	3588	6050

For $k=2$ in Table 2, the respective sequences are

Table 4. Iterated partial sums of the Pell sequence.

r \ n	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
0	1	2	5	12	29	70	169	408	985	2378	5741	13,860
1	1	3	8	20	49	119	288	696	1681	4059	9800	23,660
2	1	4	12	32	81	200	488	1184	2865	6924	16,724	40,384
3	1	5	17	49	130	330	818	2002	4867	11,791	28,515	68,899
4	1	6	23	72	202	532	1350	3352	8219	20,010	48,525	117,424

For $k=3$ in Table 2, the respective sequences are

Table 5. Iterated partial sums of the 3-Fibonacci sequence.

r \ n	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
0	1	3	10	33	109	360	1189	3927	12,970	42,837	141,481
1	1	4	14	47	156	516	1705	5632	18,602	61,439	202,920
2	1	5	19	66	222	738	2443	8075	26,677	88,116	291,036
3	1	6	25	91	313	1051	3494	11,569	38,246	126,362	417,398
4	1	7	32	123	436	1487	4981	16,550	54,796	181,158	598,556

For instance (Table 3): $S_{1,5}^{4)} = S_{1,1}^{3)} + S_{1,2}^{3)} + S_{1,3}^{3)} + S_{1,4}^{3)} + S_{1,5}^{3)}$
$= 1 + 4 + 11 + 25 + 51 = 92$.
Or (Table 4): $S_{2,5}^{4)} = S_{2,1}^{3)} + S_{2,2}^{3)} + S_{2,3}^{3)} + S_{2,4}^{3)} + S_{2,5}^{3)}$
$= 1 + 5 + 17 + 49 + 130 = 202$.

Theorem 1. *For $r \geq 1$:*

$$S_{k,n}^{r)} = S_{k,n-1}^{r)} + S_{k,n}^{r-1)}. \tag{5}$$

Proof. $S_{k,n}^{r)} = \sum_{j=1}^{n} S_{k,j}^{r-1)} = \sum_{j=1}^{n-1} S_{k,j}^{r-1)} + S_{k,n}^{r-1)} = S_{k,n-1}^{r)} + S_{k,n}^{r-1)}$. $\square$

For instance (Table 2):

$$\begin{aligned} S_{k,4}^{3)} &= k^3+3k^2+8k+13, \\ S_{k,5}^{2)} &= k^4+2k^3+6k^2+8k+9, \\ S_{k,4}^{3)}+S_{k,5}^{2)} &= k^4+3k^3+9k^2+16k+22 = S_{k,5}^{3)}. \end{aligned}$$

Theorem 2. *Sequences $S_{k,n}^{r)}$ verify the recurrence relation*

$$S_{k,n+1}^{r)} = k\,S_{k,n}^{r)} + S_{k,n-1}^{r)} + \binom{n+r-2}{n-1}. \tag{6}$$

with initial conditions $S_{k,1}^{r)}=1$ and $S_{k,2}^{r)}=k+r$.

Proof. By induction.

For $r=0$, $S_{k,n}^{0)}=F_{k,n}$ (Table 1). Formula (6) holds because $\binom{n-2}{n-1}=0$ and $S_{k,n}^{0)}$ are reduced to the k-Fibonacci numbers.

Let us suppose Formula (6) is true until $S_{k,n+1}^{r)}$ and $S_{k,n}^{r+1)}$. Then, from Equation (5),

$$\begin{aligned} S_{k,n+1}^{r)} &= S_{k,n}^{r)} + S_{k,n+1}^{r-1)} \\ &= k\,S_{k,n-1}^{r)} + S_{k,n-2}^{r)} + \binom{n+r-3}{n-2} \\ &\quad + k\,S_{k,n}^{r-1)} + S_{k,n-1}^{r-1)} + \binom{n+r-3}{n-1} \\ &= k\left(S_{k,n-1}^{r)} + S_{k,n}^{r-1)}\right) + \left(S_{k,n-2}^{r)} + S_{k,n-1}^{r-1)}\right) \\ &\quad + \left(\binom{n+r-3}{n-2} + \binom{n+r-3}{n-1}\right) \\ &= k\,S_{k,n}^{r)} + S_{k,n-1}^{r)} + \binom{n+r-2}{n-1}. \end{aligned}$$

□

For instance (Table 2), for $n=4$ and $r=3$:

$$\begin{aligned} S_{k,4}^{3)} &= k^3+3k^2+8k+13, \\ S_{k,3}^{3)} &= k^2+3k+7, \\ k\,S_{k,4}^{3)} + S_{k,3}^{3)} + \binom{6}{4} &= k^4+3k^3+9k^2+16k+22 = S_{k,5}^{3)}. \end{aligned}$$

Thus, sequence $S_{k,n}^{r)}$ can be found directly, by applying this formula, and it is not necessary to use the previous sequences.

3. Relation between the Partial Sums and the k-Fibonacci Numbers

In this section, we study the relation between the iterated partial sums of the k-Fibonacci sequences and the k-Fibonacci numbers.

Next, a lemma will be used for the following formulas.

Lemma 1.

$$\sum_{i=1}^{n} F_{k,a+i} = \frac{1}{k}\left(F_{k,a+n+1} + F_{k,a+n} - (F_{k,a+1} + F_{k,a})\right). \tag{7}$$

Proof. Since $\sum_{j=0}^{n} F_{k,j} = \frac{1}{k}(F_{k,n+1} + F_{k,n} - 1)$ and $\sum_{i=1}^{n} F_{k,a+i} = \sum_{i=0}^{n+a} F_{k,i} - \sum_{i=0}^{a} F_{k,i}$, the conclusion follows. □

For the classical Fibonacci sequence ($k = 1$), $\sum_{i=0}^{n} F_{a+i} = F_{a+n+2} - F_{a+1}$.

Corollary 1. *For $r = 2$,*

$$S^{2)}_{k,n} = \frac{1}{k^2}(F_{k,n+2} + 2F_{k,n+1} + F_{k,n} - (k\,n + k + 2)). \tag{8}$$

Proof. $S^{1)}_{k,n} = \sum_{j=1}^{n} S^{0)}_{k,j} = \sum_{j=1}^{n} F_{k,j} = \frac{1}{k}(F_{k,n+1} + F_{k,n} - 1)$. Therefore, by (7),

$$\begin{aligned}
S^{2)}_{k,n} &= \sum_{j=1}^{n} S^{1)}_{k,j} = \frac{1}{k}\sum_{j=1}^{n}\Big(F_{k,j+1} + F_{k,j} - 1\Big) \\
\sum_{j=1}^{n} F_{k,j+1} &= \frac{1}{k}(F_{k,n+2} + F_{k,n+1} - (k+1)) \\
\sum_{j=1}^{n} F_{k,j} &= \frac{1}{k}(F_{k,n+1} + F_{k,n} - 1) \\
S^{2)}_{k,n} &= \frac{1}{k^2}(F_{k,n+2} + 2F_{k,n+1} + F_{k,n} - (k\,n + k + 2)).
\end{aligned}$$

□

In particular, for $k = 1$: $S^{2)}_{1,n} = F_{n+4} - 3 - n$.

Identity (8) may be written as $S^{2)}_{k,n} = \frac{1}{k^2}\sum_{i=0}^{2}\binom{2}{i}(F_{k,n+2-i} - F_{k,2-i}) - \frac{n}{k}$.

Corollary 2. *For $r = 3$,*

$$\begin{aligned}
S^{3)}_{k,n} &= \frac{1}{k^3}\Big(F_{k,n+3} + 3F_{k,n+2} + 3F_{k,n+1} + F_{k,n} - (k^2 + 4k + 3)\Big) \\
&\quad -\frac{n}{2k^2}(k\,n + 3k + 4).
\end{aligned} \tag{9}$$

Proof. Taking into account the formulas for $S^{1)}_{k,n}$ and $S^{2)}_{k,n}$, (7) and (8),

$$\begin{aligned}
S^{3)}_{k,n} &= \sum_{j=1}^{n} S^{2)}_{k,j} \\
&= \frac{1}{k^2}\left(\sum_{j=1}^{n} F_{k,j+2} + 2\sum_{j=1}^{n} F_{k,j+1} + \sum_{j=1}^{n} F_{k,j} - k\sum_{j=1}^{n} j - (k+2)\sum_{j=1}^{n} 1\right). \\
\sum_{j=1}^{n} F_{k,j+2} &= S^{1)}_{k,n} + (F_{k,n+1} + F_{k,n+2}) - (F_{k,2} + F_{k,1}), \\
\sum_{j=1}^{n} F_{k,j+1} &= S^{1)}_{k,n} + F_{k,n+1} - F_{k,1}, \\
\sum_{j=1}^{n} F_{k,j} &= S^{1)}_{k,n},
\end{aligned}$$

$$\begin{aligned}
S_{k,n}^{3)} &= \frac{1}{k^2}\Big(4S_{k,n}^{1)} + 3F_{k,n+1} + F_{k,n+2} - F_{k,2} - 3F_{k,1} \\
&\quad - \frac{n}{2}(k\,n + 3k + 4)\Big) \\
&= \frac{1}{k^2}\left(4\frac{1}{k}(F_{k,n+1} + F_{k,n} - 1) + 3F_{k,n+1} + F_{k,n+2} - k - 3\right) \\
&= \frac{1}{k^3}\Big(F_{k,n+3} + 3F_{k,n+2} + 3F_{k,n+1} + F_{k,n} - (k^2 + 3k + 4)\Big) \\
&\quad - \frac{n}{2k^2}(k\,n + 3k + 4).
\end{aligned}$$

□

For the classical Fibonacci sequence ($k = 1$), $S_{1,n}^{3)} = F_{n+6} - 2(n^2 + n + 4)$.

Corollary 3. *For $r = 4$,*

$$\begin{aligned}
S_{k,n}^{4)} &= \frac{1}{k^4}\sum_{i=0}^{4}\binom{4}{i}(F_{k,n+4-i} - F_{k,4-i}) \\
&\quad - \frac{n}{6k^3}\Big((n^2 + 6n + 11)k^2 + (6n + 24)k + 24\Big).
\end{aligned}$$

Proof. Taking into account (8) and (4),

$$\begin{aligned}
S_{k,n}^{4)} &= \sum_{j=1}^{n} S_{k,j}^{3)} \\
&= \frac{1}{k^3}\left(\sum_{j=1}^{n} F_{k,j+3} + 3\sum_{j=1}^{n} F_{k,j+2} + 3\sum_{j=1}^{n} F_{k,j+1} + \sum_{j=1}^{n} F_{k,j}\right. \\
&\quad \left. -(k^2 + 3k + 4)\sum_{j=1}^{n} 1\right) - \frac{3k+4}{2k^2}\sum_{j=1}^{n} j - \frac{1}{2k}\sum_{j=1}^{n} j^2, \\
\sum_{j=1}^{n} F_{k,j+3} &= \frac{1}{k}(F_{k,n+4} + F_{k,n+3} - (F_{k,4} + F_{k,3})), \\
\sum_{j=1}^{n} F_{k,j+2} &= \frac{1}{k}(F_{k,n+3} + F_{k,n+2} - (F_{k,3} + F_{k,2})), \\
\sum_{j=1}^{n} F_{k,j+1} &= \frac{1}{k}(F_{k,n+2} + F_{k,n+1} - (F_{k,2} + F_{k,1})), \\
\sum_{j=1}^{n} F_{k,j} &= \frac{1}{k}(F_{k,n+1} + F_{k,n} - (F_{k,1} + F_{k,0})), \\
S_{k,n}^{4)} &= \frac{1}{k^4}(F_{k,n+4} + 4F_{k,n+3} + 6F_{k,n+2} + 4F_{k,n+1} + F_{k,n} \\
&\quad -(F_{k,4} + 4F_{k,3} + 6F_{k,2} + 4F_{k,1} + F_{k,0})) - \frac{k^2 + 3k + 4}{k^3}n \\
&\quad - \frac{3k+4}{4k^2}n(n+1) - \frac{1}{12k}n(n+1)(2n+1) \\
&= \frac{1}{k^4}\sum_{i=0}^{4}\binom{4}{i}(F_{k,n+4-i} - F_{k,4-i}) \\
&\quad - \frac{n}{6k^3}\Big((n^2 + 6n + 11)k^2 + (6n + 24)k + 24\Big).
\end{aligned}$$

□

For the classical Fibonacci sequence ($k = 1$), $S^{4)}_{1,n} = F_{n+8} - 21 - \frac{n}{6}(n^2 + 12n + 59)$.
In short:

$$
\begin{aligned}
S^{0)}_{k,n} &= F_{k,n}, \\
S^{1)}_{k,n} &= \frac{1}{k}(F_{k,n+1} + F_{k,n} - 1), \\
S^{2)}_{k,n} &= \frac{1}{k^2}(2F_{k,n+1} + 2F_{k,n} - 2 + k\,F_{k,n+1} - k - n), \\
S^{3)}_{k,n} &= \frac{1}{k^3}\Big(F_{k,n+3} + 3F_{k,n+2} + 3F_{k,n+1} + F_{k,n} - (k^2 + 3k + 4)\Big) \\
&\quad - \frac{n}{2k^2}(k\,n + 3k + 4), \\
S^{4)}_{k,n} &= \frac{1}{k^4}\sum_{i=0}^{4}\binom{4}{i}(F_{k,n+4-i} - F_{k,4-i}) \\
&\quad - \frac{n}{6k^3}\Big((n^2 + 6n + 11)k^2 + (6n + 24)k + 24\Big).
\end{aligned}
$$

4. Conclusions

In this paper, we have found the sequence of partial sums of the k-Fibonacci sequence, say, $S_{k,n} = \sum_{j=1}^{n} F_{k,j}$, and then the sequence of partial sums of this new sequence, $S^{2)}_{k,n} = \sum_{j=1}^{n} S_{k,j}$, and so on. The iterated partial sums of k-Fibonacci numbers have been given as functions of k-Fibonacci numbers, in powers of k, and in a recursive way.
Finally, a formula to find the first terms of these sequences from the k-Fibonacci numbers themselves has also been proved.

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