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A RESULT IN ELEMENTARY MATHEMATICAL ANALYSIS

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ABSTRACT.

The following result is proved, highlighting didactic and paedagogical aspects: For every Riemann integrable function $f : [0,1] \longrightarrow \mathbf{R}^+$, the function $A(h) = \int_0^1 |h-f(x)|\,dx$, for h running between the extrema of f, is concave upwards.

RESUMEN.

Se prueba, con una motivación elemental y propósitos didácticos, que toda función integrable Riemann $f : [0,1] \longrightarrow \mathbf{R}^+$, satisface que $A(h) = \int_0^1 |h-f(x)|\,dx$ es cóncava. La función $A(h)$ está definida en el intervalo dado por los extremos de $f(x)$ en $[0,1]$.

1. A SIMPLE RESULT OF THE INTEGRAL CALCULUS.

Let us consider a continuous monotonically decreasing function f defined on $[0,1]$. We shall compute the following integral:

$$A(h) = \int_0^1 |h-f(x)|\,dx$$

where $h \in [f(0),f(1)]$. Under the hypotheses satisfied by f, it is an injective map. We write $F = f^{-1}$ for its inverse function. Let us now write, for every $h \in]f(1),f(0)[$, $A(h)$ in the form:

$$A(h) = \int_0^{F(h)} (f(x)-h)\,dx + \int_{F(h)}^1 (h-f(x))\,dx.$$

By applying the derivative of an integral with respect to a parameter under the supplementary hypothesis that f (and, consequently, also F in $(f(0),f(1))$) is differentiable in the open interval $(0,1)$, we obtain:

$$\frac{dA}{dh} = -\int_0^{F(h)} dx + [f(F(h))-h]F'(h) + \int_{F(h)}^1 dx - [h-f(F(h))]F'(h) = 1-2F(h) + 2[f(F(h))-h]F'(h)$$

From $F = f^{-1}$ it follows that for every $h \in [f(1),f(0)]$, $f(F(h)) = h$ holds, so $\frac{dA}{dh} = 1-2F(h)$, thus implying that the derivative is zero only for $h = f(\frac{1}{2})$. Now we use the Inverse Function Theorem in order to build the second derivative of $A(h)$ for a local analysis:

$$\frac{d^2A}{dh^2} = -2F'(h) = -\frac{2}{f'(F(h))}$$

We observe that $f'(x) < 0$ for every $x \in (0,1)$, so the second derivative of $A(h)$ is always positive. Thus our function is concave upwards and has no other extrema than a unique minimum. We sum up the above ideas in the following proposition:

PROPOSITION 1:

"Let $f(x)$ be a strictly monotonous decreasing continuous function defined in the interval $[0,1]$, and let $f(x)$ be differentiable in $(0,1)$. Then the function $A(h) = \int_0^1 |h - f(x)|\, dx$, defined in the interval $[f(1), f(0)]$, is concave upwards and attains a unique minimum at $f(\frac{1}{2})$".

2. A GENERAL RESULT.

Mathematical reasoning tends to deal with general ideas, so we can try a new problem by asking whether a result analogous to Proposition 1 could be obtained for a wider class of functions. We shall impose only one condition on the function $f(x)$: It must be Riemann integrable on $[0,1]$. Our function need not be decreasing, but for the sake of simplicity we shall suppose it is strictly positive.

Let us consider the function $A(h) = \int_0^1 |h - f(x)|\, dx$, defined on $[k,K]$, where k, K are the extrema of f in the interval $[0,1]$.

In order to solve this general problem, we shall define the concept of the *decreasingly ordered* function f^* associated to a real function $f:[0,1] \longrightarrow \mathbf{R}^+$. The idea is to construct a new, but decreasing, function $f^*:[0,1] \longrightarrow \Re^+$ having the same range of values as f.

DEFINITION:

The decreasingly ordered function f^* associated to f is defined by the relationship $f^*(\Omega(x)) = f(x)$, where $\Omega: [0,1] \longrightarrow [0,1]$ is a one-to-one function having the following property: $\Omega(x) < \Omega(y)$ if $f(x) > f(y)$, and if $f(x) = f(y)$, then $\Omega(x) < \Omega(y)$ if $x < y$.

For a step function $E(x)$ taking constant values $K_1, K_2, \ldots, K_n$ on the consecutive subintervals of $[0,1]$ defined by a set of points $0 = x_0 < x_1 < \ldots < x_{n-1} < x_n = 1$, $E^*(x)$ is readily defined by shuffling the intervals in such a way that the values K_j appear decreasingly ordered. For the step function E^* the subintervals will be defined by a new point set denoted by $0 = y_0 < y_1 < \ldots < y_{n-1} < y_n = 1$, and we shall rename the values K_j as $k_1 \ge k_2 \ge k_3 \ge \ldots \ge k_{n-1} \ge k_n$.

The crucial point is the rather obvious fact that a step function $E(x)$ and its decreasingly ordered function $E^*(x)$ *define the same function* $A(h)$. For a step function, the graph of $A(h)$ is a piecewise affine continuous polygonal defined on $[k,K]$, with corner-like points at the values k_j. Let us compute the value $A(h)$ for a step function $E(x)$. Due to the above remark the computation is done on $E^*(x)$:

For every h in the open interval (k,K) there exists some j between 1 and $n-1$ such that $k_j \ge h \ge k_{j+1}$. The value of $A(h)$ is easily computed:

$$A(h) = \sum_{r=1}^{j} (k_r - h)(y_r - y_{r-1}) + \sum_{s=j+1}^{n} (h - k_s)(y_s - y_{s-1}) =$$

$$= \left(-\sum_{r=1}^{j} (y_r - y_{r-1}) + \sum_{s=j+1}^{n} (y_s - y_{s-1}) \right) h + \ldots =$$

$$= (1 - 2y_j)h + \dots$$

so, for $k_j < h < k_{j+1}$, the slope of the polygonal is $1 - 2y_j$. As h runs from k to K, the y_j run from 1 to 0, and the sign of the slope changes from positive to negative while the y values run through $1/2$. Thus we have proved the following lemma:

LEMMA .

"If $E(x)$ is a step function defined on $[0,1]$, then the function $A(h) = \int_0^1 |h - E(x)|\,dx$ is concave upwards".

It is easy to see that $A(h)$ attains a minimum at the first value $k_j > 1/2$ if no value in the partition is equal to $1/2$. In this particular case, the graph of $A(h)$ will show a horizontal stretch over the interval $k_j < h < k_{j+1}$

Now let us consider any Riemann integrable function $f(x)$ on the interval $[0,1]$. We build a family of approximating step functions $E_i(x)$ yielding the corresponding concave functions $A_i(h)$. In the limit, by using the elementary properties of integrable functions, we obtain $A_i(h) \longrightarrow A(h)$.

Now we must prove that the limit of concave functions is also concave: Let $y = r_i(h)$ be the equation of the line joining the points $(k, A_i(k))$ and $(K, A_i(K))$. For every h it is true that $A_i(h) \leq r_i(h)$, and in the limit we have that $A(h)$ is concave upwards. Thus we have proved the following proposition:

PROPOSITION 2:

"Let $f(x)$ be a positive integrable function on the interval $[0,1]$, whose extrema are k and K. Then the function $A(h) = \int_0^1 |h - f(x)|\,dx$, defined on $[k,K]$, is concave upwards".

Uniqueness of the minimum is not guaranteed unless more restrictive conditions are imposed on the function (See Proposition 1 and the remark about the above Lemma). In any case the topography of the graph of A(h) shows some type of minimum in the neighbourhood of $x = 1/2$.

3. TWO REMARKS ON MOTIVATION.

3.1. ON THE ORIGIN OF THE PROBLEM.

The motivation for this elementary study appeared when trying to solve the following exercise for classroom use. See [1]:

"The upper half of a circle is given, and a chord parallel to the diameter is drawn at variable height h between the perpendiculars at both ends of the diameter . Find the minimum of the area bounded by the circle and the chord". A standard solution is the following, which inspired the result in Proposition 1 above:

Due to symmetry with respect to the y-axis, let us use the equation of the part of the upper half circle, $f(x)=\sqrt{1-x^2}$, only in the first orthant. For any height h in the interval $[0,1]$ there exists a unique value of $x=F(h)$, i.e. $f(x)$ is injective, which is equivalent to $F=f^{-1}$. By taking the derivative of the integral with respect to the parameter h we obtain:

$$\frac{dA}{dh} = -\int_0^{F(h)} dx + [f(F(h))-h]F'(h) + \int_{F(h)}^{1} dx - [h - f(F(h))]F'(h) = 1 - 2F(h) + 2[f(F(h))-h]F'(h)$$

From $F = f^{-1}$ we have that $h \in [f(1), f(0)] \Rightarrow f(F(h)) = h$, so $\frac{dA}{dh} = 1 - 2F(h)$. Thus the derivative is zero for $h = f(\frac{1}{2})$. Now we apply the Inverse Function Theorem in order to find the second derivative of $A(h)$:

$$\frac{d^2A}{dh^2} = -2F'(h) = -\frac{2}{f'(F(h))}$$

We observe that f is monotonically decreasing, so $f'(x) < 0$ for every $x \in (0,1)$. Thus the second derivative of A is always positive. Therefore our function is concave upwards and its extremum is unique and is a minimum.

As we have seen, the function $f(x) = \sqrt{1-x^2}$ in the interval $[0,1]$ is the right model for the exercise, so a unique minimum exists for the chord drawn at a height $h = \sqrt{1-(\frac{1}{2})^2} = \frac{\sqrt{3}}{2}$.

Other elementary solutions based on trigonometric identitites are also available, but do not open such an interesting field of ideas.

3.2. SOME NEW PROSPECTIVE IDEAS.

For any given function f, there exists an infinite number of functions yielding the same f^*. This can be shown easily by considering the case of a step function. More than that, from $E(x)$ one can build $E^*(x)$ algorithmically. There exist also trivial cases, like the one of any increasing function on $[0,1]$, where f^* has a graph symmetric to that of f with respect to the line $x = \frac{1}{2}$.

Loosely speaking, f^* will always have some (remote) similarity with Lebesgue's *singular function* whose graph is very popular in books on Fractal Geometry under the name of *devil's staircase*. The question is: Is it possible to build, either analytically or algorithmically, f^* after a given f ?

4. REFERENCE.

[1] Anonymous (1993) Entrance examinations for the Independent University of Moscow, *Notices of the American Mathematical Society, 40(2)*, 138-139.

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