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Distribution Theory
Research Paper

The arctan family of distributions: New results with applications

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Abstract

In this paper, we explore the family of arctan transformation of a distribution function. We get some general properties such as those related to the right tail and scale transformation, among others. The results obtained are used to generalize the Pareto Type II (also known as Lomax) distribution, giving us a distribution with a long right-tail that admits the zero value in its support. We show some properties and provide closed-form expressions for the raw moments, the quantile function, the tail value at risk, and other analytical forms that can be helpful in financial and actuarial settings, such as the limited expected value, the mean excess function, and the integrated tail distribution. We also show three numerical illustrations including health expenditure for outpatients, automobile insurance claim size and to see how the new model works as compared to other distributions used in the applied statistical literature.

Keywords: Actuarial · arctan function · Claim size · Income · Pareto type II Distribution · Right tail

Mathematics Subject Classification: 62E10 · 62F10 · 62P05 · 62P25.

1. Introduction

Gómez-Déniz and Calderín-Ojeda (2015a) introduced a mechanism to add a shape parameter to a parent distribution by using the arctan trigonometric transformation of this parent model. They studied the case where the parent distribution was replaced by the classical Pareto cumulative distribution function (CDF). Due to this transformation, results for this new model were obtained including very nice properties. The case where the parent survival function (SF) is the exponential distribution was studied in Calderín-Ojeda et al. (2016). The discrete case was investigated in Gómez–Déniz et al. (2019), obtaining a generalization of the geometric distribution. Furthermore, this transformation was also used in income distribution by Gómez-Déniz (2016), getting the corresponding Lorenz and Leimkhuler curves.

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After showing the general properties of this family, one of its particular cases is investigated, the arctan Pareto type II distribution. We derive some essential properties, which are simple consequences of the properties of the general family. The flexibility of this distribution is illustrated by applying it to three empirical data sets and comparing the results to previously used distributions.

An apparent reason for generalizing a standard or parent distribution is that the generalized form provides greater flexibility as compared to the parent distribution. For example, consider the problem of determining a suitable model for a population for which it is desired to make a inference. A common way to carry out this task is to use a general model that includes a simpler one as a particular case or limit. After fitting both models, the one that yields the best inference is chosen. Experience indicates that the general model produces better results than the simplest model.

The rest of the paper is structured as follows. Properties of the family of the arctan transformation of a CDF are studied in Section 2. Section 3 is devoted to the specific subject of dealing with the Pareto type II distribution. Numerical applications are considered in Section 4, and finally, Section 5 concludes the work.

2. The arctan transformation and general properties

In this section we firstly illustrate the general procedure to derive the arctan family of distributions. Next, we present some relevant properties of this family. Finally we show that the arctan family of probability distributions can be ordered in terms of the usual stochastic order.

2.1 General methodology

Gómez-Déniz and Calderín-Ojeda (2015b) provided a method to add a scale parameter to a distribution (parent distribution), obtaining a more flexible distribution than the parent model. To make this paper self-contained, we reproduce here this methodology which is based on the $\tan^{-1}$ (arctan) transformation of the parent distribution.

The half-Cauchy distribution (Jacob and Jayakumar, 2012) truncated at $\alpha > 0$ has probability density function (PDF) given by

$$f(y) = \frac{1}{\tan^{-1}\alpha}\frac{1}{1+y^2}, \quad 0 < y < \alpha. \tag{2.1}$$

In the latter expression, $\tan^{-1}$ is the inverse of the circular tangent function. Let us consider now the transformation $y = \alpha\bar{F}_\Theta(x)$, where $\bar{F}_\Theta$ is the SF of a random variable X with support in $[a, b]$, whereas a and b can be finite or non-finite, and Θ is a parameter or vector of parameters. Then, the corresponding PDF of the random variable X obtained from Equation (2.1) results

$$f_{\Theta,\alpha}(x) = \frac{1}{\tan^{-1}\alpha}\frac{\alpha f_\Theta(x)}{1+[\alpha\bar{F}_\Theta(x)]^2}, \tag{2.2}$$

for $a \le x \le b$ and $\alpha > 0$. The SF of X, which is obtained from Equation (2.2) by integration, is stated as

$$\bar{F}_{\Theta,\alpha}(x) = \frac{\tan^{-1}(\alpha\bar{F}_\Theta(x))}{\tan^{-1}\alpha}. \tag{2.3}$$

Furthermore, it is simple to see that Equations (2.2) and (2.3) are proper PDF and SF, respectively, when the support of the parameter α is extended to $(-\infty,\infty)$ except for zero. In this case, we get that $\bar{F}_{\Theta,\alpha}(x)=\bar{F}_{\Theta,-\alpha}(x)$. Additionally, by taking in Equation (2.3) limit when the parameter α tends to zero and applying the L'Hospital rule, it is straightforward to derive that the parent SF, $\bar{F}_{\Theta}$, is obtained as a particular case, that is, $\bar{F}_{\Theta,\alpha}(x)\to\bar{F}_{\Theta}(x)$ when $\alpha\to 0$. Thus, this methodology can be considered a mechanism to add a scale parameter to a parent SF and, therefore, a mechanism to obtain a more flexible SF. In particular, the case where $\bar{F}$ is replaced by the CDF of the classical Pareto distribution was considered in Gómez-Déniz and Calderín-Ojeda (2015b) and Gómez-Déniz (2016) and the case where the parent SF is the classical exponential distribution was studied in Calderín-Ojeda et al. (2016). The discrete case was studied in Gómez–Déniz et al. (2019) obtaining a generalization of the classical geometric distribution. Also, in actuarial statistics, the arctan transformation was first used in Gómez-Déniz and Calderín-Ojeda (2015a).

2.2 Properties

The quantile function is easy to derive from Equation (2.3) and it is given by

$$x_\gamma = F^{-1}_{\Theta,\alpha}\left(1-\alpha^{-1}\tan(\bar{\gamma}\tan^{-1}\alpha)\right), \tag{2.4}$$

where $\bar{\gamma}=1-\gamma$, $0<\gamma<1$ and F^{-1} is the inverse of the CDF F. In particular, the median is expressed as

$$x_{0.5} = F^{-1}_{\Theta,\alpha}\left(1-\alpha^{-1}\tan((0.5)\tan^{-1}\alpha)\right),$$

Proposition 2.1 Suppose that the parent SF depends on a vector of parameters $\Theta=(\theta_1,\ldots,\theta_s)$ satisfying $\bar{F}_{\Theta}(x/k)=\bar{F}_{\Theta_1}(x)$, being Θ_1 a vector of parameters for which the parameter j, for some $j\in\{1,\ldots,s\}$ is a scale or rate transformation of θ, with rate or scale value $k>0$. Then, the arctan distribution preserves also the same transformation.

Proof By denoting $Y=k\,X$, and denoting the SF of Y as $\bar{F}^Y_{\Theta,\alpha}$, we have that

$$\bar{F}^Y_{\Theta,\alpha}(y)=\bar{F}^Y_{\Theta,\alpha}(kx)=\bar{F}_{\Theta,\alpha}(x/k)=\frac{\tan^{-1}(\alpha\bar{F}_{\Theta}(x/k))}{\tan^{-1}\alpha}=\frac{\tan^{-1}(\alpha\bar{F}_{\Theta_1}(x))}{\tan^{-1}\alpha}=\bar{F}_{\Theta_1,\alpha}(x),$$

where in the last equality we have used the assumption that $\bar{F}_{\Theta}(x/k)=\bar{F}_{\Theta_1}(x)$. $\square$

To illustrate Proposition 2.1, consider the exponential distribution with mean $\Theta=1/\lambda$ and $\lambda>0$. Then, it is simple to verify that

$$P(kX>x)=P(X>x/k)=\exp(-x/(\lambda k)),$$

that is, the random variable kX follows an exponential distribution with parameter $\Theta_1=1/(\lambda k)$. Now, the arc transformation of the exponential distribution has SF given by

$$\bar{F}_{\Theta,\alpha}(x)=\frac{\tan^{-1}(\alpha\exp(-x/\lambda))}{\tan^{-1}\alpha},$$

which satisfies $F_{\Theta,\alpha}(x/k)=F_{\Theta_1,\alpha}(x)$ as it can be verified in a simple way.

It is already known that any probability distribution, that is specified through its CDF $F(x)$ on the real line, is heavy right-tailed (Rolski et al., 1999) if $\limsup_{x\to\infty}(-\log(\bar{F}(x)/x)) = 0$. Observe that $-\log(\bar{F}(x)$ is the hazard function of $F(x)$. Next, a result shows that, under mild condition, the family of SF provided in Equation (3.8) is a heavy-tailed distribution.

Proposition 2.2 Suppose that the PDF of the parent distribution in the family stated in Equation (2.2) satisfies that

$$\limsup_{x\to\infty} f_\Theta(x) = 0. \tag{2.5}$$

Then, the CDF $F_{\Theta,\alpha}$ of the family defined in Equation (3.8) is a heavy-tailed distribution.

Proof We have that

$$\begin{aligned}\limsup_{x\to\infty} \frac{1}{x}\log \bar{F}_{\Theta,\alpha}(x) &= -\frac{1}{\tan^{-1}\alpha}\limsup_{x\to\infty}\frac{\log(\tan^{-1}(\alpha\bar{F}_\Theta(x)))}{x}\\ &= \frac{\alpha}{\tan^{-1}\alpha}\limsup_{x\to\infty}\frac{f_\Theta(x)}{1+\alpha^2[\bar{F}_\Theta(x)]^2} = 0,\end{aligned}$$

after applying the L'Hospital rule. The fact that $\limsup_{x\to\infty}\bar{F}_\Theta(x) = 0$ and the assumption that $\limsup_{x\to\infty} f_\Theta(x) = 0$ conduct to the result.□

In this case, the distribution fails to possess any positive exponential moment, that is, $\int \exp(sx)\mathrm{d}F(x) = \infty$ for all $s > 0$ (Foss et al., 2011, Ch. 1, p. 2). Distributions of this type have moment generating function $M_F(s) = \infty$, for all $s > 0$, as occurs, for example, with the lognormal distribution. As a consequence of the last result, we have the following corollary.

Corollary 2.3 It is verified that $\limsup_{x\to\infty}\exp(sx)\bar{F}_{\Theta,\alpha}(x) = \infty$, for $s > 0$.

Proof This is a direct consequence of Proposition 2.2.□

An important issue in extreme value theory is the regular variation (Bingham, 1987 and Konstantinides, 2018). This is, a fexible description of the variation of some function according to the polynomial form of the type $x^{-\delta} + o(x^{-\delta})$, $\delta > 0$. This concept is formalized in the following definition.

Definition 2.4 A CDF (measurable function) is called regular varying at infinity with index $-\delta$ if it holds

$$\limsup_{x\to\infty}\frac{\bar{F}(\tau x)}{\bar{F}(x)} = \tau^{-\delta},$$

where $\tau > 0$ and the parameter $\delta \geq 0$ is called the tail index.

The next result establishes that if the SF of the parent distribution stated in Equation (3.8) is a regular variation Lebesgue measure, then the SF given in Equation (3.8) is also a regular variation Lebesgue measure.

Proposition 2.5 Let $\bar{F}_\Theta(x)$ ibe a regular variation Lebesgue measure. Then, the SF given in Equation (2.3) is also a SF with regularly varying tails.

Proof Consider the SF given in Equation (2.3). Then, we have

$$\limsup_{x\to\infty} \frac{\bar{F}_{\Theta,\alpha}(\tau x)}{\bar{F}_{\Theta,\alpha}(x)} = \limsup_{x\to\infty} \frac{f_\Theta(\tau x)}{f_\Theta(x)} \frac{1+\alpha^2[\bar{F}_\Theta(x)]^2}{1+\alpha^2[\bar{F}_\Theta(\tau x)]^2} = \tau^{-(\Theta+1)},$$

after applying the L'Hospital rule. The fact that $\limsup_{x\to\infty} \bar{F}_{\Theta,\alpha}(\tau x) = \limsup_{x\to\infty} \bar{F}_{\Theta,\alpha}(x) = 0$ and that $f_\Theta(\tau x)/f_\Theta(x) \to \tau^\Theta$ when $x \to \infty$, conduct to the result.□

In actuarial setting and also into the individual and collective risk models the practitioner is usually interested in the random variable $S_n = \sum_{i=1}^n X_i$ for $n \geq 1$. Although in practice, its PDF is difficult or impossible to calculate, we can approximate its probabilities by using the following Corollary, which is an immediate consequence of Proposition 2.5 (Jessen and Mikosch, 2006).

Corollary 2.6 Let $X_1, \ldots, X_n$ be independent identically distributed random variables with common SF given by Equation (2.3) and $S_n = \sum_{i=1}^n X_i,\ n \geq 1$. Then, we get

$$P(S_n > x) \sim P(X > x) \quad \text{as} \quad x \to \infty. \tag{2.6}$$

Therefore, if $P_n = \max_{i=1,\ldots,n} X_i$, for $n \geq 1$, we have that

$$P(S_n > x) \sim nP(X > x) \sim P(P_n > x).$$

This means that, for large x, the event $\{S_n > x\}$ is due to the event $\{P_n > x\}$. Therefore, exceedences of high thresholds by the sum S_n are due to the exceedence of this threshold by the largest value in the sample.

As Jessen and Mikosch (2006) pointed out, expression given in Equation (2.6) can be taken as the definition of a subexponential distribution. The class of those distributions is greater than the class of regularly varying distributions. The result given in Corollary 2.6 remains valid for subexponential distributions in the sense that subexponentiality of S_n implies subexponentiality of X_1. Usually, this property is referred to as convolution root closure of subexponential distributions. More details can be viewed in Embrechts and Goldie (1980) and Embrechts and Goldie (1982).

2.3 Stochastic ordering

Next, a stochastic representation of the parameters of the given family in Equation (2.3) is studied. As it is well known, many parametric families of distributions can be stated by means of some stochastic orders according to the value of its parameters. For the general family of distributions given in Equation 2.3, it is difficult to establish an order in terms of the likelihood ratio order (Ross, 1996; Shaked and Shanthikumar, 2007). For a particular choice of the main distribution, this is possible (Gómez-Déniz and Calderín-Ojeda, 2015a). However, a weaker but also useful result may be obtained, as shown below.

Definition 2.7 Let us consider two random variables X_{Θ_1} and X_{Θ_2}, with X_{Θ_1} preceding X_{Θ_2} in the stochastic dominance sense or X_{Θ_1} being smaller than X_{Θ_2}. In this case, the notation $X_{\Theta_1} \leq_{\mathrm{ST}} X_{\Theta_2}$ is used, if and only if the CDF of X_{Θ_1} always exceeds X_{Θ_2}, that is,

$$F_{\Theta_1}(x) \geq F_{\Theta_2}(x), \quad -\infty < x < \infty,$$

where F_{Θ_1} and F_{Θ_2} are the CDFs of X_{Θ_1} and X_{Θ_2} respectively. Note that this expression is the same as

$$\bar{F}_{\Theta_1}(x) \leq \bar{F}_{\Theta_2}(x), \quad -\infty < x < \infty.$$

In the following, we provide two stochastic orderings. In the first one the order is given by fixing the shape parameter α and modifying the parameters of the parent distribution, whereas in the second one we have fixed the parameters vector of the parent distribution and changed the parameter α.

PROPOSITION 2.8 Let us consider two random variables X_1 and X_2 with CDFs $F_{\Theta_1}(x)$ and $F_{\Theta_2}(x)$, respectively such that X_1 is stochastically smaller than X_2 ($X_1 \leq_{\mathrm{ST}} X_2$), that is, $F_{\Theta_1}(x) \geq F_{\Theta_2}(x)$ for $\Theta_1 \leq \Theta_2$. Then, the arctan transformation preserves this stochastic order, that is, $F_{\Theta_1,\alpha}(x) \geq F_{\Theta_2,\alpha}(x)$.

PROOF Since the arctan function is monotone, we have that

$$\begin{aligned}
F_{\Theta_1}(x) \geq F_{\Theta_2}(x) &\Longrightarrow \alpha F_{\Theta_1}(x) \geq \alpha F_{\Theta_2}(x) \Longrightarrow \tan^{-1}(\alpha F_{\Theta_1}(x)) \geq \tan^{-1}(\alpha F_{\Theta_2}(x)) \\
&\Longrightarrow \frac{\tan^{-1}(\alpha F_{\Theta_1}(x))}{\tan^{-1}\alpha} \geq \frac{\tan^{-1}(\alpha F_{\Theta_2}(x))}{\tan^{-1}\alpha} \\
&\Longrightarrow F_{\Theta_1,\alpha}(x) \geq F_{\Theta_2,\alpha}(x).
\end{aligned}$$

Hence, the result is obtained.□

THEOREM 2.9 Let X_1 and X_2 be two random variables with PDFs $f_{\Theta,\alpha_1}(x) > 0$ and $f_{\Theta,\alpha_2}(x) > 0$ obtained from Equation (2.2), respectively. If $\alpha_1 \leq \alpha_2$, then $X_1 \leq_{\mathrm{LR}} X_2$.

PROOF Note that the ratio

$$\frac{f_{\Theta,\alpha_2}(x)}{f_{\Theta,\alpha_1}(x)} = \frac{\alpha_2 \tan^{-1}\alpha_1}{\alpha_1 \tan^{-1}\alpha_2} m_{\Theta,\alpha_1,\alpha_2}(x)$$

is non-decreasing if and only if $m'_{\Theta,\alpha_1,\alpha_2}(x) \geq 0$ for x in their support, where

$$m_{\Theta,\alpha_1,\alpha_2}(x) = \frac{1 + [\alpha_1 \bar{F}_\Theta(x)]^2}{1 + [\alpha_2 \bar{F}_\Theta(x)]^2}.$$

Some calculations show that

$$m'_{\Theta,\alpha_1,\alpha_2}(x) = \frac{2 f_\Theta(x) \bar{F}_\Theta(x)(\alpha_2^2 - \alpha_1^2) m_{\Theta,\alpha_1,\alpha_2}(x)}{(1 + [\alpha_1 \bar{F}_\Theta(x)]^2)(1 + [\alpha_2 \bar{F}_\Theta(x)]^2)}.$$

Now, taking into account that $\alpha_1 \leq \alpha_2$, then $m_{\Theta,\alpha_1,\alpha_2}(x) \geq 0$ and the result holds.□

We have now the following corollary.

COROLLARY 2.10 Let X_1 and X_2 be two random variables with PDFs $f_{\Theta,\alpha_1}(x) > 0$ and $f_{\Theta,\alpha_2}(x) > 0$ obtained from Equation (2.2), respectively and hazard rates $h_{\Theta,\alpha_1}(x)$ and $h_{\Theta,\alpha_2}(x)$, being $h_{\Theta,\alpha}(x) = f_{\Theta,\alpha}(x)/\bar{F}_{\Theta,\alpha}(x)$, respectively. If $\alpha_1 \leq \alpha_2$ then,

(i) $\mathrm{E}(X_1^k) \leq \mathrm{E}(X_2^k)$ for all $k > 0$,
(ii) $h_{\Theta,\alpha_1}(x) \leq h_{\Theta,\alpha_2}(x)$ for all x in their support.

Proof It is well-known (Shaked and Shanthikumar, 2007) that

$$X_1 \leq_{\mathrm{LR}} X_2 \Longrightarrow X_1 \leq_{\mathrm{HR}} X_2 \Longrightarrow X_1 \leq_{\mathrm{ST}} X_2. \tag{2.7}$$

Therefore, (i) follows from Theorem 2.9 and Equation (2.7) by taking into account that $X_1 \leq_{\mathrm{ST}} X_2$ holds if and only if

$$\mathrm{E}\left[\Phi\left(X_1\right)\right] \leq \mathrm{E}\left[\Phi\left(X_2\right)\right] \text{ for all non-decreasing function } \Phi.$$

Similarly, (ii) follows by combining Theorem 2.9 and Equation (2.7). Thus, in consequence, if $\alpha_1 \leq \alpha_2$ we have that $\bar{F}_{\Theta,\alpha_1}(x) \leq \bar{F}_{\Theta,\alpha_2}(x)$.

3. The Lomax arctan distribution

In this section, we firstly introduce the Lomax arctan distribution (LAT hereafter) and derive some of its more relevant statistical and financial properties.

3.1 Specific model

A particular case of the Pareto Type II distribution is considered here. This distribution is essentially a classical Pareto distribution modified to get that the support begins at zero. As it is known, this distribution is widely employed as a model in business, economics, actuarial science, queueing theory, and internet traffic modeling, among others. Its SF is given by

$$\bar{F}_{\Theta}(x) = \left(\frac{\lambda}{\lambda + x}\right)^{\sigma}, \quad x \geq 0, \tag{3.8}$$

(Fisk, 1961; Suárez-Espinosa et al., 2018) which is a particular case of the Champernowne distribution (Champrenowne, 1952) and obviously is a scale transformation of the classical Pareto distribution (Arnold, 1983). A Lomax regression model with varying precision parameter was recently presented in Melo et al. (2021) In the rest of the paper we use $X \sim \mathrm{L}(\sigma, \alpha)$ to point out that X follows a Pareto Type II distribution with the PDF given in Equation (3.9).

An excellent property of this distribution, apart from having a very tractable SF, is a fascinating preservation property. That is, if $X \sim \mathrm{L}(\sigma, \lambda)$, then the random variable $kX \sim \mathrm{L}(\sigma, k\lambda)$, for $k > 0$. This property is very useful in economics and actuarial fields when dealing with inflation. The PDF, derived from Equation (3.8), results

$$f_{\Theta}(x) = \frac{\sigma\lambda^{\sigma}}{(x+\lambda)^{\sigma+1}}, \quad x \geq 0, \;\; \sigma > 0, \;\; \lambda > 0. \tag{3.9}$$

One of the advantages of working with the SF given in Equation (3.8) is the possibility of dealing with data that includes the zero value, the mode of the distribution. This is impossible for most classical continuous distributions, such as the gamma and the inverse Gaussian distribution. Nevertheless, the distribution has limited flexibility for adapting to empirical data whose modal value is not located at zero. To get a more flexible distribution, we consider here the $\tan^{-1}$ transformation of the Pareto Type II distribution. The resulting distribution, Pareto Type II arctan distribution, it is obtained by applying the Equation

(2.3) to Equation (3.8) to get the SF given by

$$\bar{F}_{\Theta,\alpha}(x) = \frac{\tan^{-1}(\alpha(1+x/\lambda)^{-\sigma})}{\tan^{-1}\alpha}. \tag{3.10}$$

Its PDF results

$$f_{\Theta,\alpha}(x) = \frac{\alpha\sigma}{\lambda\tan^{-1}\alpha}\frac{(1+x/\lambda)^{-\sigma-1}}{1+\alpha^2(1+x/\lambda)^{-2\sigma}}. \tag{3.11}$$

Figure 1 shows several graphs of the PDF given in Equation (3.11) for different values of its parameters. It is noted that when the scale parameter $\alpha < 1$ or the shape parameter $\sigma \leq 1$, the mode of the distribution is located at 0, and for values larger than one, the modal value moves to the right. Observe that the larger are the value of the parameters, the greater is the modal value.

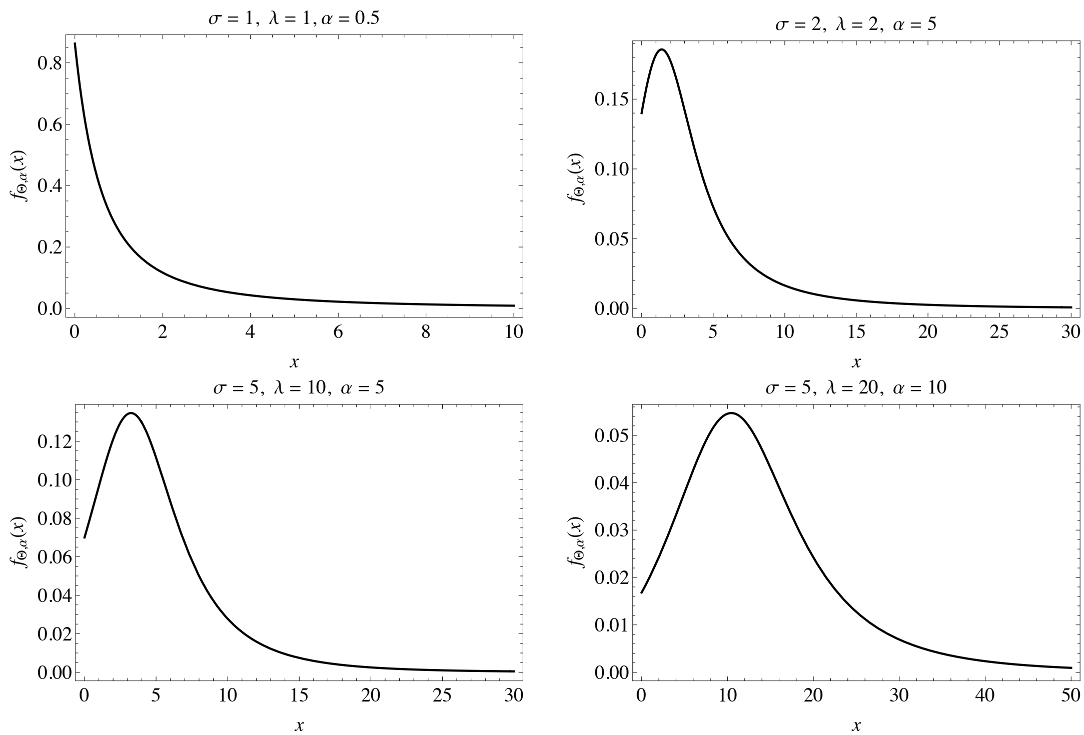


Figure 1. PDF the LAT distribution for selected values of parameters σ, λ and α

Since this distribution is a scale transformation of the Pareto arctan distribution studied in Gómez-Déniz and Calderín-Ojeda (2015a), we can easily obtain its row moments that are given by

$$\mathrm{E}(X^r) = \frac{\alpha\sigma\lambda^r}{\tan^{-1}\alpha}\sum_{j=0}^{r}\frac{(-1)^j}{\sigma-r+j}\binom{r}{j}\,{}_2F_1\left(1, \frac{\sigma-r+j}{2\sigma}; \frac{3\sigma-r+j}{2\sigma}; -\alpha^2\right),$$

where ${}_2F_1$ is the hypergeometric function defined as

$${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\mathrm{d}t.$$

In particular, the mean takes the form

$$\mu = \mathrm{E}(X) = \frac{\alpha\lambda\sigma}{(\sigma-1)\tan^{-1}\alpha}\, {}_2F_1\left(1, \frac{\sigma-1}{2\sigma}; \frac{3\sigma-1}{2\sigma}; -\alpha^2\right) - \lambda, \quad \sigma > 1. \tag{3.12}$$

From Equation (2.4), the quantile function x_γ is simply derived as

$$x_\gamma = \lambda\left\{\left[\frac{1}{\alpha}\tan\left(\bar{\gamma}\tan^{-1}\alpha\right)\right]^{-1/\sigma} - 1\right\}, \tag{3.13}$$

and from Equation (3.13), the median can be easily obtained.

The mode, which can be obtained by differentiating Equation (3.11) with respect to the variable x, is expressed as

$$x_{Mo} = \lambda\left[\left(\frac{\alpha^2(\sigma-1)}{1+\sigma}\right)^{(2\sigma)^{-1}} - 1\right].$$

Then, the hazard rate function for the LAT distribution, $h_{\Theta,\alpha}(x) = f_{\Theta,\alpha}(x)/\bar{F}_{\Theta,\alpha}(x)$, which is obtained from Equations (3.10) and (3.11), has been plotted for the same values of parameters as considered in the previous Figure. This is shown in Figure 2. It can be observed that the hazard rate function has a variety of shapes. For example, for values of $\alpha < 1$, the hazard rate function is monotonically decreasing and for values of the scale parameters α and the shape parameter σ and scale parameter λ the function is firstly increasing and then decreasing.

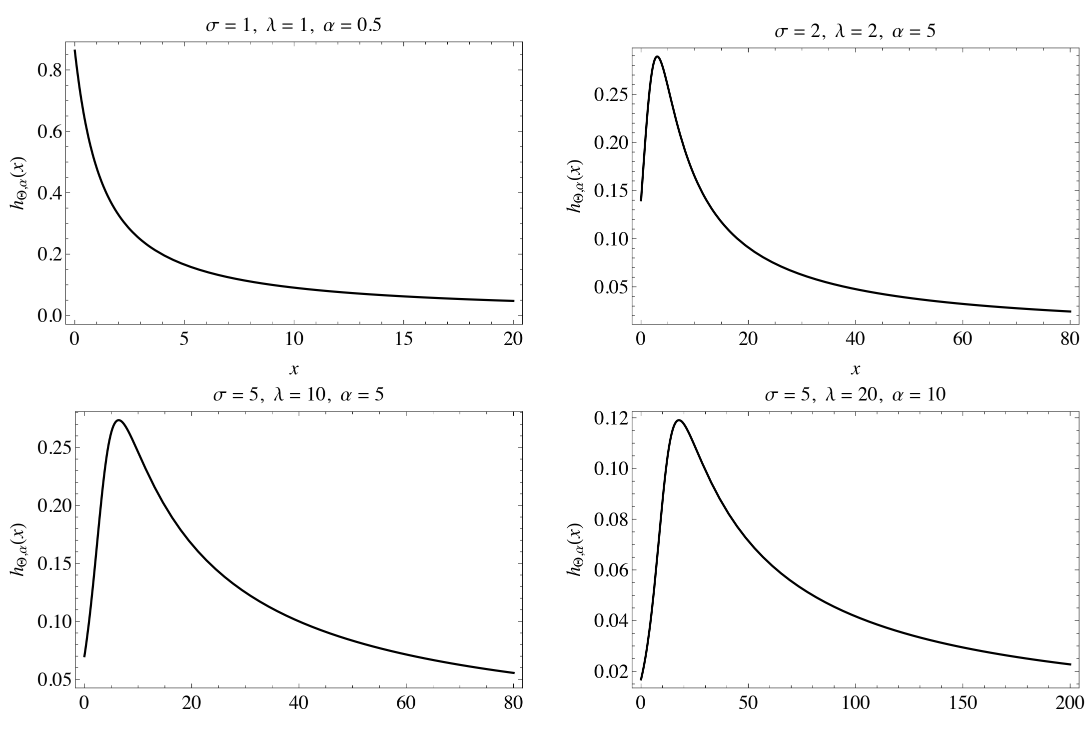


Figure 2. Failure rate function of LAT distribution for selected values of parameters σ, λ and α

We now provide some properties which are consequences of the results obtained in the previous section.

Proposition 3.1 If $X \sim \mathrm{LAT}(\sigma, \lambda, \alpha)$ then $kX \sim \mathrm{LAT}(\sigma, k\lambda, \alpha)$.

Proof It is a direct consequence of Proposition 2.1.$\square$

Proposition 3.2 The CDF $F_{\Theta,\alpha}(x)$ of the family stated in Equation (3.10), that is, the LAT distribution is a heavy-tailed distribution.

Proof It is a direct consequence of applying Proposition 2.2 having into account that the PDF given in Equation (3.9) satisfies Equation (2.5).

Proposition 3.3 The SF given in Equation (3.10) is a SF with regularly varying tails.

Proof It is a consequence of the result provided in Proposition 2.5, having into account that the SF given in Equation (3.8) verifies

$$\lim_{x\to\infty}\sup \frac{\bar{F}_\Theta(\tau x)}{\bar{F}_\Theta(x)} = \tau^{-\sigma}.$$

Now, because $\sigma > 0$, we have the result.$\square$

3.2 Further properties

We provide here some other properties which can be helpful in financial and actuarial fields. Let the random variable

$$Z = X \wedge \omega = \begin{cases} X, \; X < \omega, \\ \omega, \;\; X \geq \omega, \end{cases}$$

which is an amount used in excess of loss reinsurance context with excess level $\omega > 0$. Insurance companies widely use this tool to reduce the amount paid on larger claims. Its expected value, $\mathrm{E}(X \wedge \omega)$, is referred to as the limited expected value in insurance context. Obviously, it is a right-censored variable for which it is easy to see (Hogg and Klugman, 1984; Boland, 2007) that can be computed as

$$\mathrm{E}(X \wedge \omega) = \mathrm{E}[\min(X, \omega)] = \int_0^\omega x\, f(x)\, \mathrm{d}x + \omega\, \bar{F}(\omega). \tag{3.14}$$

Furthermore, it represents the expected amount per claim retained by the insured on a policy with a fixed amount deductible of ω. Thus, defining the expected dollar (or other monetary units) saving per incident when a deductible is imposed (Klugman et al., 2008, Ch. 3).

For the LAT distribution, the limited expected value given by Equation (3.14) is expressed as

$$\mathrm{E}(X \wedge \omega) = (\omega + \lambda)\bar{F}_{\Theta,\alpha}(\omega) - \lambda + H^1_{\Theta,\alpha}(\omega) - H^2_{\Theta,\alpha}(\omega), \tag{3.15}$$

where

$$H^1_{\Theta,\alpha}(\omega) = \frac{\lambda+\omega}{\bar{F}(\omega)}\, {}_2F_1\left(1, \frac{1+\sigma}{2\sigma}; \frac{1}{2}\left(3+\frac{1}{\sigma}\right); (\alpha\bar{F}_{\Theta,\alpha}(\omega))^{-2}\right), \tag{3.16}$$

$$H^2_{\Theta,\alpha}(\omega) = \lambda\, {}_2F_1\left(1, \frac{1+\sigma}{2\sigma}; \frac{1}{2}\left(3+\frac{1}{\sigma}\right); (\alpha\bar{F}_{\Theta,\alpha}(\omega))^{-2}\right), \tag{3.17}$$

which can be obtained also by using a scale transformation of the classical Pareto distribution (Gómez-Déniz and Calderín-Ojeda, 2015a).

The value at risk (VaR) is defined as the amount of capital required to ensure that the insurer does not become insolvent with a high degree of certainty. The VaR of a random variable X which follows the LAT distribution is the $100q$th quantile and therefore coincides with Equation (3.13).

It is known that the use of the VaR is questionable due to the lack of subadditivity. For that reason, the expected loss given that the loss exceeds the $100q$th quantile of the distribution of X, that is, the tail value at risk (TVaR), is considered. Then, if X follows a LAT distribution, for any quantile q, the tail value at risk, can be obtained again by a scale transformation of the TVaR of the classical Pareto distribution and is given by

$$\begin{aligned}\mathrm{TVaR}(X;q) &= \frac{1}{1-q}\int_q^1 \mathrm{VaR}(x;q)\,\mathrm{d}q = \frac{\alpha\lambda\sigma}{\bar{q}(\sigma-1)\tan^{-1}\alpha}\left[\frac{\tan(\bar{q}\tan^{-1}\alpha)}{\alpha}\right]^{1-1/\sigma} \\ &\quad\times\ {}_2F_1\left(1, \frac{\sigma-1}{2\sigma}; \frac{3}{2}-\frac{1}{2\sigma}; -\tan^2(\bar{q}\tan^{-1}\alpha)\right) - \lambda.\end{aligned}$$

The integrated tail distribution (also known as equilibrium distribution) is an important distribution that often appears in insurance and many other applied probability models.

Let $\bar{F}$ be the SF given in Equation (3.10). Then, the integrated tail distribution of F (for instance, Klüppelberg, 1988 and Yang, 2004) is defined as $F^I(x) = (1/\mathrm{E}(X))\int_0^x \bar{F}(y)\,\mathrm{d}y$. For the distribution proposed in this work, as proven in the following result, the integrated tail distribution can be written as a closed-form expression and given by

$$F^I_{\Theta,\alpha}(x) = \frac{1}{\mu}\left[(x+\lambda)\bar{F}_{\Theta,\alpha}(x) - \lambda\right] + \frac{\sigma}{\alpha\mu(\sigma+1)\tan^{-1}\alpha}\left[H^1_{\Theta,\alpha}(x) - H^2_{\Theta,\alpha}(x)\right], \tag{3.18}$$

where $H^j_{\Theta,\alpha}(x)$, for $j=1,2$, are given in Equations (3.16) and (3.17), respectively, whereas $\bar{F}_{\Theta,\alpha}$ and μ are defined in Equations (3.10) and (3.12), respectively. Under the classical model (Embrechts and Veraverbeke, 1982; Yang, 2004) and assuming a positive security loading, ρ, for the claim size distributions with regularly varying tails we have that, by using Equation (3.18), it is possible to obtain an approximation of the probability of ruin, $\Psi(u)$, when $u\to\infty$. In this case, the asymptotic approximation of the ruin function is stated as $\Psi(u)\sim(1/\rho)\bar{F}^I(u)$, for $u\to\infty$, where $\bar{F}^I(u) = 1-F^I(u)$.

The failure rate of the integrated tail distribution, which is expressed as $\gamma_I(x) = \bar{F}(x)/\int_x^\infty \bar{F}(y)\,\mathrm{d}y$, is also obtained in closed-form. Furthermore, the reciprocal of γ_I is the mean residual life that can be easily derived. For a claim amount random variable X, the mean excess function (also known as the conditional mean exceedence) is the expected payment per claim for a policy with a fixed amount deductible of $x>0$, where claims with amounts less than or equal to x are wholly ignored. Then, we have that

$$e(x) = \mathrm{E}(X-x|X>x) = \frac{1}{\bar{F}(x)}\int_x^\infty \bar{F}(u)\,\mathrm{d}u. \tag{3.19}$$

This function is also essential in an actuarial setting, when we deal with reinsurance (Albrecher et al., 2017). If X is a lifetime, as in demography or reliability, Equation (3.19) is recognized as the mean residual lifetime. The following result gives the mean excess function of the LAT distribution in a closed-form expression.

PROPOSITION 3.4 The mean excess function of the LAT distribution is given by

$$e_{\Theta,\alpha}(x) = \frac{1}{\bar{F}_{\Theta,\alpha}(x)} \left[\mu + \lambda - \frac{\sigma(H^1_{\Theta,\alpha}(x) - H^2_{\Theta,\alpha}(x))}{\alpha(1+\sigma)\tan^{-1}\alpha} \right] - (x+\lambda), \tag{3.20}$$

where $H^j_{\Theta,\alpha}(x)$, for $j = 1, 2$, are given in Equations (3.16) and (3.17), respectively, whereas $\bar{F}_{\Theta,\alpha}$ and μ are given in Equations (3.10) and (3.12), respectively.

PROOF Using the expression

$$e(x) = \frac{\mathrm{E}(X) - \mathrm{E}(X \wedge x)}{\bar{F}(x)},$$

which relates the mean excess function given in Equation (3.19) with the limited expected value function (Hogg and Klugman, 1984, p. 59), the result follows by using and Equations (3.12), (3.10), (3.15) and a some little algebra.□

Figure 3 shows the mean residual life function given in Equation (3.20) for special cases of parameters. It can be seen that this function can be increasing, decreasing, unimodal or anti-unimodal.

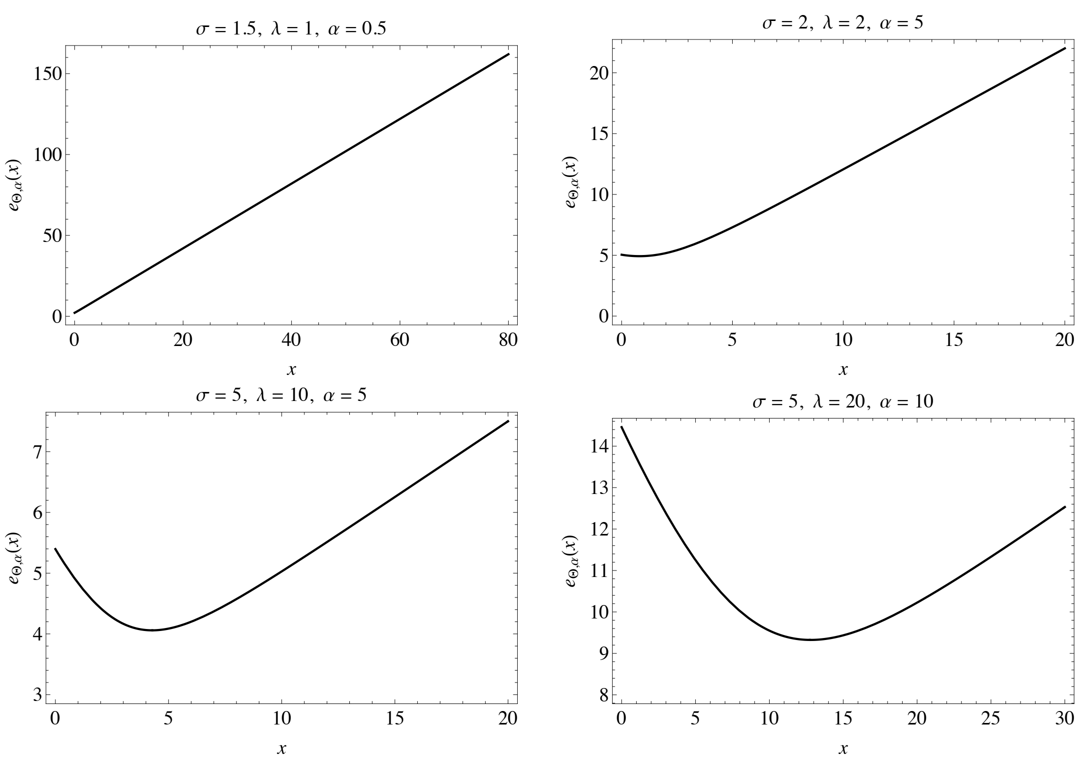


Figure 3. Mean residual life function of LAT distribution for selected values of parameters

4. Illustrative examples

In this section, we examine the practical performance of the LAT distribution in three examples that can be found in the personal web page of Professor E. Frees Frees (2010) (examples 1 and 3) and another one available in Klugman (1991) (example 2). All the data used in this work are displayed in Appendix.

The parameters are estimated using `WinRats` (Brooks, 2009) for examples 1 and 2, while `Mathematica` v.12.0 (Ruskeepaa, 2009) is used for example 3. The values of the supplied tests and the p-values were obtained using the `R` software. Graphical plots have been made employing `Mathematica` and `R`. All calculations were carried out on Windows-supported computers with an `i7-7700 CPU@3.60GHz` processor with response times for all examples standard.

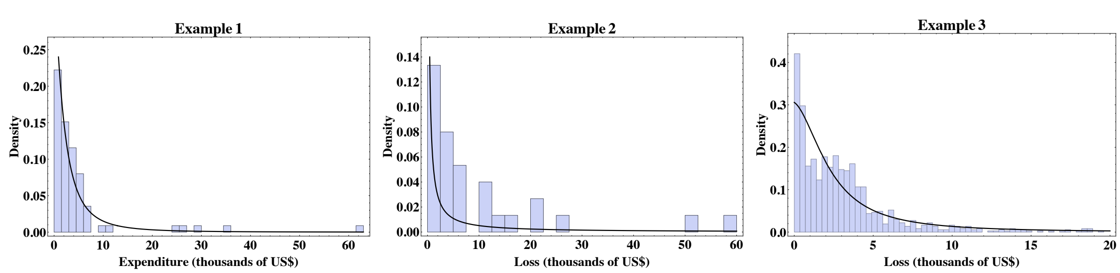
4.1 Example 1

The data were obtained from the Medical Expenditure Panel Survey (MEPS), conducted by the U.S. Agency of Health Research and Quality. MEPS is a probability survey that provides nationally representative estimates of health care use, expenditures, sources of payment, and insurance coverage for the U.S. civilian population. The variable of interest consist of amounts of expenditures for outpatient (EXPENDOP) visits. In the first row of Table 1, we report the descriptive statistics of the empirical data that seems to be unimodal and positively skewed. In Figure 4(al), it is displayed the histogram of the empirical data and the PDF plot corresponding to Example 1.

The log-likelihood function together with the normal equations, which provide the maximum likelihood estimates, are shown in Appendix of this article.

Table 1. Descriptive statistics of the data sets used in the indicated example.

Example	n	Mean	Standard deviation	Minimum	Maximum
1	75	4.95594	9.32897	0	62.8111
2	30	9.54	14.16	0	59
3	1091	5.3262	16.1746	0.005	273.604



(a) (b) (c)

Figure 4. Empirical histograms and PDF plots for examples 1 (a), 2 (b) and 3 (c).

We compare the LAT distribution introduced in this work with other competing models proposed in the literature that have the capacity to incorporate zero observations in the sample. As a benchmark, we consider the classical exponential distribution with mean $1/\lambda$, for $\lambda > 0$, the Lomax distribution and the generalized exponential distribution due to

Marshall and Olkin (1997), with SF given by

$$\bar{F}(x) = \frac{\lambda \exp(-\sigma x)}{1 - \bar{\lambda} \exp(-\sigma x)}, \quad x \geq 0, \sigma > 0, \lambda > 0,$$

and $\bar{\lambda} = 1 - \lambda$.

In Table 2 are exhibited the parameter estimates together with their standard errors (in brackets) for the four models considered. It can be seen that the LAT distribution provides the best fit to data in terms of the two measures of model selection examined, negative of the maximum of the log-likelihood function (NLL) and Akaike information criterion (AIC). Model selection was also assessed from a practical perspective using the Kolmogorov-Smirnov (KS) and the Crámer-von Mises (CM) goodness-of-fit tests to quantify the distance between the empirical CDF (ECDF) constructed from the data and the ones generated from the fitted models. Let $\widehat{F}$ denote the CDF of the fitted model, the original data by $x_1, \ldots, x_N$ and the ordered data in increasing magnitude by $x_{(1)}, \ldots, x_{(N)}$. Then the expressions of the KS and CM statistics are defined as:

(i) Kolmogorov-Smirnov test statistic: $D = \max(D^+, D^-)$, where

$$D^+ = \max_{1 \leq j \leq N} \left| \frac{j}{N} - \widehat{F}(x_{(j)}) \right|, \; D^- = \max_{1 \leq j \leq N} \left| \widehat{F}(x_{(j)}) - \frac{j-1}{N} \right|.$$

(ii) Crámer-von Mises test statistic:

$$W^2 = \sum_{j=1}^{N} \left[\widehat{F}(x_{(j)}) - \frac{2j-1}{2N} \right]^2 + \frac{1}{12N}.$$

Results on the goodness of fit of the four parametric models considered are also presented in last four rows of Table 2. Note that the LAT distribution yields lower values for both test statistics and it is not rejected for both tests as judged by the corresponding p-values.

Table 2. Example 1. Parameter estimates for the exponential (E), Lomax (L), generalized exponential (GE) and LAT distributions via maximum likelihood estimation. Standard errors are provided in parenthesis and p-values for the KS and CM tests between brackets.

Parameter	E	L	GE	LAT
$\widehat{\lambda}$	0.202	0.181	0.129	4.624
	(0.023)	(0.085)	(0.077)	(0.209)
$\widehat{\sigma}$		2.106	0.058	1.991
		(0.718)	(0.029)	(0.048)
$\widehat{\alpha}$				-0.581
				(0.238)
n	75	75	75	75
NLL	195.044	182.833	183.653	182.811
AIC	392.088	369.666	371.306	371.622
KS	0.157	0.747	0.107	0.077
	[0.10]	[< 0.001]	[0.652]	[0.97]
CM	0.752	0.227	0.191	0.127
	[0.007]	[0.237]	[0.293]	[0.460]

4.2 Example 2

In the second example, we use data that can be found in Appendix of Klugman (1991). In particular, we employ the data set 2 ,where the loss value for the first year in the 30 first classes have been taken. The second row of Table 1 shows the descriptive statistics of this second data set and in the middle panel of Figure 4 are illustrated the ECDF and the smooth CDF for the second example. In Table 3, we report the parameter estimates together with their standard errors (in brackets) for four of the models previously considered. Once again, it can be seen that the LAT distribution provides a marginal best fit data in terms of the negative of the NLL. However, when the AIC is considered, the GE distribution provides a slightly better fit to this dataset. Model selection was also assessed via KS and CM goodness-of-fit tests to quantify the distance between the ECDF constructed from the data and the ones generated from the fitted models. As judged by these tests, the LAT distribution is not rejected at usual significance levels.

Table 5 reports empirical and theoretical limited expected values for the LAT distribution with different values of the policy limit x, using the parameter estimates calculated for the dataset given in Example 2 and the expression defined in Equation (3.15). Note that for large values of x, that is, when x tends to infinity, the limited lev approaches to the mean of the distribution. Nevertheless, as in this case $\alpha < 1$, the mean does not exist.

Table 3. Example 2. Parameter estimates for the exponential (E), Lomax (L), generalized exponential (GE) and LAT distributions via maximum likelihood estimation. Standard errors are provided in parenthesis and p-values for the KS and CM tests between brackets.

Parameter	E	L	GE	LAT
$\widehat{\lambda}$	0.105	0.188	0.129	3.07×10^{-6}
	(0.019)	(0.174)	(0.123)	(0.003)
$\widehat{\sigma}$		1.340	0.034	0.176
		(0.751)	(0.027)	(0.016)
$\widehat{\alpha}$				-7.639
				(0.116)
n	30	30	30	30
NLL	97.644	93.726	93.034	63.969
AIC	197.288	191.452	190.068	133.938
KS	0.300	0.629	0.200	0.264
	[0.071]	[8.2×10^{-7}]	[0.586]	[0.134]

4.3 Example 3

The third dataset deals with automobile bodily injury claims data from the Insurance Research Council (IRC), a division of the American Institute for Chartered Property Casualty Underwriters and the Insurance Institute of America. The data, collected in 2002, contain information on demographic information about the claimant, attorney involvement and the economic loss (in thousands of US\$). We consider a sample of 1091 losses from a single state. The third row of Table 1 reports descriptive statistics of this third data set, and in the bottom panel of Figure 4, the ECDF and the smooth CDF are displayed for the third example.

In Table 5, we report the parameter estimates together with their standard errors (in brackets) for the LAT and GE distributions and two models traditionally used to explain income data the lognormal (LO) distribution with parameters $\lambda \in (-\infty, \infty)$ and $\sigma > 0$ and

Table 4. Empirical and theoretical limited expected value for the LAT distribution and different values of the policy limit x for the second example dataset.

Policy limit (x)	Empirical	Fitted
0	0.00	0.00
2	1.50	0.99
4	2.70	1.82
6	3.57	2.58
8	4.27	3.31
10	4.94	4.02
12	5.47	4.70
14	5.90	5.37
16	6.27	6.02
18	6.60	6.66
20	6.94	7.29
22	7.20	7.91
24	7.40	8.53
26	7.60	9.13
28	7.77	9.73
30	7.90	10.32
32	8.04	10.90
34	8.17	11.48
36	8.30	12.05
38	8.44	12.62
40	8.57	13.18
42	8.70	13.74
44	8.84	14.30
46	8.97	14.85
48	9.10	15.40
50	9.24	15.94

the Singh-Maddala (SM) distribution with SF given by

$$\bar{F}(x) = \left[1 + \left(\frac{x}{\sigma}\right)^{\lambda}\right]^{-\alpha}, \quad x \geq 0, \sigma > 0, \lambda > 0, \alpha > 0. \tag{4.21}$$

Observe that the special case $\sigma = 1$ reduces Equation (4.21) to the Burr type XII distribution studied by Rezac et al. (2015). Once again, it can be seen that the LAT distribution provides the best fit to data in terms of the two measures of model selection examined, negative of the NLL and AIC. Model selection was also assessed via KS and CM goodness-of-fit tests to quantify the distance between the ECDF constructed from the data and the CDFs generated from the fitted models. As judged by the measures of model selection, the LAT distribution provides the best fit to the data. Moreover, although the LAT distribution is rejected in terms of the KS test and the CM test at the usual significance levels, the value of the test statistics are the lower among all the models considered.

Table 5. Example 3. Parameter estimates for the generalized exponential (GE), lognormal (LO), Singh-Maddala (SM) and LAT distributions via maximum likelihood estimation. Standard errors are provided in brackets and p-values between brackets.

Parameter	GE	LO	SM	LAT
$\widehat{\lambda}$	0.051	0.620	1.103	2.283
	(0.013)	(0.085)	(0.044)	(0.042)
$\widehat{\sigma}$	0.025	1.445	3.672	1.667
	(0.006)	(0.045)	(0.031)	(0.566)
$\widehat{\alpha}$			1.643	-1.709
			(0.188)	(0.514)
n	1091	1091	1091	1091
NLL	2637.87	2626.74	2601.69	2598.20
AIC	5279.74	5257.48	5209.38	5202.39
KS	0.089	0.093	0.062	0.052
	[< 0.001]	[< 0.001]	[< 0.001]	[0.005]
CM	2.715	2.446	1.200	1.002
	[< 0.001]	[< 0.001]	[< 0.001]	[0.004]

5. Conclusions, limitations, and future research

In this paper, we derive several properties related to the family of arctan transformation of a survival function, mainly those connected with the right tail of the distribution. After this, we introduced the arctan transformation of the Pareto Type II distribution, a scale transformation of the classical Pareto distribution. This is a model for non-negative continuous random variables, including the zero value in its support. We have provided in closed-form expression the raw moment, quantile function, the tail value at risk, and other functions which can be helpful in the financial and actuarial field, such as the integrated tail distribution, the limited expected value, and the mean excess function.

The performance of this new family of distributions has been illustrated by using three different data sets. The first one was associated with the expenditures for outpatients; the second one was related to the third party automobile insurance claims; and the final example considered automobile injury claims. Numerical results showed that the Lomax arctan distribution is helpful to explain heavy-tailed empirical data. However, although this distribution is able to capture the presence of zeros in the data, if the proportion of zeros is too high, the model has a worse performance in relation to the other models, as it is shown in the second example.

Further analysis of this probabilistic family remains as a topic for future studies. In this regard, investigation of the multivariate version of the arctan transformation is a topic that deserves to be examined in upcoming works in depth in upcoming works.

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Appendix

Let us assume that $X_1, \ldots, X_n$ is a random sample selected from the distribution given in Equation (3.11), with their observations denoted by $x_1, \ldots, x_n$. The corresponding likelihood function is given by

$$\begin{aligned}\ell(\Theta, \alpha; \tilde{x}) = n(\log(\alpha) + \log(\sigma) - \log(\lambda) - \log(\tan^{-1}(\alpha)) - (\sigma + 1)\sum_{i=1}^{n} \log(1 + x_i/\lambda) \\ - \sum_{i=1}^{n} \log\left(1 + \alpha^2(1 + x_i/\lambda)^{-2\sigma}\right). \qquad (5.22)\end{aligned}$$

The normal equations obtained from Equation (5.22) are stated as

$$\frac{\partial \ell(\Theta, \alpha; \tilde{x})}{\partial \sigma} = \frac{n}{\sigma} - \sum_{i=1}^{n} \log(1 + x_i/\lambda) + 2\alpha^2 \sum_{i=1}^{n} \frac{(1 + x_i/\lambda)^{-2\sigma} \log(1 + x_i/\lambda)}{1 + \alpha^2(1 + x_i/\lambda)^{-2\sigma}} = 0, \qquad (5.23)$$

$$\frac{\partial \ell(\Theta,\alpha;\tilde{x})}{\partial \lambda} = -\frac{n}{\lambda} + \frac{\sigma+1}{\lambda^2}\sum_{i=1}^{n}\frac{x_i}{1+x_i/\lambda} + \frac{2\sigma\alpha^2}{\lambda^2}\sum_{i=1}^{n}\frac{x_i(1+x_i/\lambda)^{-2\sigma-1}}{1+\alpha^2(1+x_i/\lambda)^{-2\sigma}} = 0,$$

$$\frac{\partial \ell(\Theta,\alpha;\tilde{x})}{\partial \alpha} = n\left[\frac{1}{\alpha} - \frac{1}{(1+\alpha^2)\tan^{-1}\alpha}\right] - 2\alpha\sum_{i=1}^{n}\frac{(1+x_i/\lambda)^{-2\sigma}}{1+\alpha^2(1+x_i/\lambda)^{-2\sigma}} = 0, \quad (5.24)$$

from which we can get the maximum likelihood estimates of the parameters by a numerical method such as Newton-Raphson. On taking the second partial derivatives of Equations (5.23)-(5.24), the Fisher information matrix $\mathcal{I}(\Theta,\alpha)$ can be obtained by taking the expectations of minus the second derivatives. The inverse of the matrix provides the variances for the maximum likelihood estimators.

Table 6. Data for example 1.

1.4683	35.9342	0	0	7.24614	4.62498	3.80673	1.95896
4.62158	3.72445	0.87823	0.28698	1.80114	6.73978	3.69576	0.06081
3.56721	24.2046	5.82075	6.46576	2.53495	0.69315	1.68874	0.82613
5.32987	3.46299	1.68822	0.03755	6.47876	2.58618	9.5353	0.54148
1.95018	1.18143	3.74168	0.77534	2.88031	2.40923	3.00777	0.36825
0.59158	0.05376	5.8413	0.17115	1.78891	0.47681	0.68236	
62.8111	0.12816	4.18265	5.37448	1.99109	3.76849	0.31383	
1.45670	3.44599	1.19869	2.56363	2.01848	2.56077	5.63908	
5.99927	3.08074	29.1859	5.20015	4.06117	2.13937	2.85663	
26.0717	0.11700	1.83426	0	10.8975	0.19800	4.37083	

Table 7. Data for example 2.

1	3	5	0	15	27	0	3	0	11
6	20	0	13	11	4	22	0	3	50
10	4	7	1	59	2	1	3	5	0

Information for authors

The editorial board of the Chilean Journal of Statistics (ChJS) is seeking papers, which will be refereed. We encourage the authors to submit a PDF electronic version of the manuscript in a free format to the Editors-in-Chief of the ChJS (E-mail: `chilean.journal.of.statistics@gmail.com`). Submitted manuscripts must be written in English and contain the name and affiliation of each author followed by a leading abstract and keywords. The authors must include a"cover letter" presenting their manuscript and mentioning: "We confirm that this manuscript has been read and approved by all named authors. In addition, we declare that the manuscript is original and it is not being published or submitted for publication elsewhere".

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Cook, R.D., 1997. Local influence. In Kotz, S., Read, C.B., and Banks, D.L. (Eds.), Encyclopedia of Statistical Sciences, Vol. 1., Wiley, New York, pp. 380-385.

Rukhin, A.L., 2009. Identities for negative moments of quadratic forms in normal variables. Statistics and Probability Letters, 79, 1004-1007.

Stein, M.L., 1999. Statistical Interpolation of Spatial Data: Some Theory for Kriging. Springer, New York.

Tsay, R.S., Peña, D., and Pankratz, A.E., 2000. Outliers in multivariate time series. Biometrika, 87, 789-804.

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