

## Forum (Robustness Analysis)

### Robustness Analysis: An Information-Based Perspective

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#### 1. Introduction

Decision-making problems are *–not only but essentially–* information problems. Such information tells us about levels of aspiration or satisfaction, goals, criteria, among others. If all the right information is available at the right moment and the desired alternative is reachable, there is no decision to make. Otherwise the decision-making process comprises discovering, investigating, interpreting and adapting knowledge from what is envisaged until the moment when the right alternative to choose comes along.

Likewise, robustness problems are decision-making problems and therefore information problems. Searching for robustness implies coping with ignorance. Sometimes such ignorance could be reduced, sometimes not. It is important to notice that, even in those cases where the ignorance could be reduced, on occasions the high price of the additional knowledge could not justify the gain in information. Thus, the natural option is to deal with ignorance instead of reducing it.

For instance, in robust design we search a system configuration or setting that is able to resist variation in its input without a significant loss of quality –like a major deviation from a target value– in its output. Why do we search such a design? Clearly because we consider that the resulting loss of quality entails regrettable consequences; otherwise

we could change our minds and accept the output's variability. In other words, Decision-Makers (DM) are supposed to define when the output is undesirable or moreover unacceptable.

Independently of the DM's criteria and no matter what the system is (a method, a decision, an optimal solution, a physical system –see Vincke 2003 for a discussion–); we always can find situations when the usefulness of a system could be sensitively affected due to input's uncertainty. For example, a decision could be no longer appropriate if the scenarios where the decision is based on change. The same thing might occur with, e.g., an optimal alternative. Once implemented, this alternative could experiment a considerable loss of optimality in the presence of uncertain values of its decision variable. This diversity of systems and situations explains why the concept of robustness meets so many realizations as those presented in earlier issues of the present forum as well as in a large number of publications in the field (see e.g. Sayin, 2005).

A natural question that derives from this wide horizon of robustness formulations is what concept should an analyst use and why? The answer to this question logically depends on the information available and naturally on the kind of system into consideration. The labour of the analysts is, in consequence, a two-stage task. First the particular formulation of robustness should be drawn from the most generic idea of robustness; then the analyst has to decide how to solve the problem.

As the authors have a special interest in optimisation and particularly in evolutionary optimisation, the following discussion is focused on robust solutions.

#### 2. Information-Based Robustness Analysis

##### 2.1. A Generic Robustness Formulation

Perhaps the most common and very general formulation of robustness states that a system is robust when its output is insensitive to small variation of its input (Sayin, 2005). In optimisation it is said that a solution is robust when the value of the objective function does not change significantly when the decision vector are slightly shifted inside its neighbourhood. However, none of the above concepts define how "small" an input's variation should be or what "insensitive" means. Since the size of these qualifiers depends on DM's criteria, we argue that any analyst should investigate what information the DM may provide, among other

factors, before defining the particular formulation of robustness that is applicable to the problem under consideration.

Let us consider a general optimisation problem:

$$\begin{aligned} & \text{Opt } F(x) \\ & \text{s.t.:} \\ & G_j(x) \leq b_j, j= 1,2,\dots,m. \\ & x \in X \end{aligned} \quad (1)$$

Within an uncertain environment, it is commonly assumed that the decision vector  $x$  is exposed to a source of variability that could be represented as  $x+\delta_i$ , where the vector  $\delta_i$  is a particular realization of uncertainty phenomenon  $\delta$ . Nevertheless this assumption could be invalid in some discrete domains that only allow being represented by means of scenarios. Another concern often presented in robustness analysis is the constraint satisfaction.

For the sake of simplicity, let us focus now on those problems that might be represented by considering  $\delta$ . Evidently, the DM will be interested in assessing the effect of the uncertainty on the output; thus the objective is transformed in some robustness indicator that has  $x,\delta$  as an argument. Besides we know that despite the size of the variation,  $\delta$  may be bounded yielding  $\delta_{\min} \leq \delta \leq \delta_{\max}$ . A vector of uncertain independent variables or parameters  $p_{\min} \leq p \leq p_{\max}$  can be defined as well. Consequently we must define the new objective considering these additional elements and their ranges of variation.

The aforementioned elements can be integrated to yield a general robustness formulation as:

$$\begin{aligned} & \text{Opt } R(F, x, \delta, p, \gamma) \\ & \text{s.t.:} \\ & G_j(x, \delta, p) \leq b_j (j= 1,2,\dots,J) \\ & x \in X \\ & F_{\min} \leq F(x, \delta, p) \leq F_{\max} \\ & \delta_{\min} \leq \delta \leq \delta_{\max} \\ & p_{\min} \leq p \leq p_{\max} \end{aligned} \quad (2)$$

where the new objective function  $R$  is the robustness measure, which is function of the value of  $F$  in the presence of uncertainty  $(\delta,p)$  and in accordance with DM's criteria. The extra parameter  $\gamma$  is necessary for some formulation as will be explained later on. The bounds  $F_{\min}$  and  $F_{\max}$  are related to the maximal and minimal value that the original function  $F(x)$  reaches over  $x, \delta$ . Such bounds could serve to state goals or

levels of attainment as well as to control the size of the output's variability.

## 2.2. Deriving Robustness Formulations: a two-stage Information-based perspective

### 1<sup>st</sup> Stage: Robustness Definition

The previous formulation has the intention of being as generic as possible in order to unify diverse concepts present in the literature. Hence, according to this perspective the first stage is a conceptual stage: how can we define robustness in terms of the given information?

The analyst should precisely define what is known and what is unknown in order to find which definition of robustness may be employed. Some questions that could be posed to help the analyst task are:

- **Domain:** What is known about the domain and what can be assumed?
  - Discrete or continuous domain?
  - It is possible to define a variable neighbourhood?
  - Are there constraints? Should they be strictly satisfied?
- **Uncertainty:** What is the uncertainty source?
  - Should the uncertainty be represented by scenarios, by a probability law, intervals...?
  - It is possible to describe  $\delta$  with a probability distribution function (PDF)? What PDF? What are the parameters? Other forms?
- **Robustness criteria:** What is the functional expression of  $R(F, x, \delta, p, \gamma)$ ?
  - Are there target values? ( $\gamma = f_{\text{target}}$ )
  - What is the DM's attitude? (risk-lover, risk-adverse)
  - It is possible to define constraint and/or goals over the output?

With the answers to these questions the analyst defines the particular robustness problem to be solved. This constitutes the first stage. Then the analyst must decide the proper method to solve it, completing the second stage.

Now let us derive some robustness problems from the generic formulation in (2). For the sake of simplicity the only source of uncertainty considered from now on is  $\delta$ . Nonetheless the analysis could be easily extended to consider vector  $p$ .

The following table summarizes some of the different cases an analyst could find.

INFORMATION	KNOWN $\delta_{\min} \leq \delta \leq \delta_{\max}$	UNKNOWN $\delta_{\min} \leq \delta \leq \delta_{\max}$
KNOWN $F_{\min} \leq F(x, \delta, p) \leq F_{\max}$	Case 1	Case 3
UNKNOWN $F_{\min} \leq F(x, \delta, p) \leq F_{\max}$	Case 2	Case 4

*Approach 1: uncertainty propagation*

Cases 1 and 2 correspond to those approaches based on uncertainty propagation. Here a description of  $\delta$  is necessary. Usually  $\delta$  is described by means of a PDF with mean zero and  $\sigma$  the standard deviation. Depending on the type of PDF,  $\delta_{\min}$ ,  $\delta_{\max}$  takes different values; e.g.  $\delta_{\min} = -\infty$ ,  $\delta_{\max} = \infty$  for a normal law or  $\delta_{\min} = -a$ ,  $\delta_{\max} = a$  where  $a$  is a vector of finite scalars, for a uniform law.

Some typical criteria employed in these cases are:

- Optimization of the expected value E(x): if no preference is expressed about the output (case 2) the first stage is completed making  $R(x, \delta) = E(x, \delta)$ , and the second stage consists in determining how the uncertainty will be propagated (Sampling -Monte Carlo, Latin Hypercube, Importance sampling (Du & Chen, 1999); Interval Arithmetic (Kolev, 1994), Probability Bounds Analysis (Ferson & Hajagos, 2004)). Notice that a typical subproblem that could arise at this stage is the comparison among intervals.

A pretty common example of case 2 is the so called Effective Function (Tsutsui & Gosh, 1997; Sørensen, 2003; Sevaux & Sørensen, 2004):

$$R(x, \delta) = F_{\text{eff}}(x) = \frac{1}{n} \sum_{i=1}^n F(x + \delta_i)$$

Taguchi's robust design principle also belongs to case 2. Here the DM establish a target value around which the deviation should be minimized, yielding:

$$R(x, \delta, \gamma = f_{\text{target}}) = \max \{ \text{dist}(F_{\min}, f_{\text{target}}), \text{dist}(F_{\max}, f_{\text{target}}) \}$$

Case 2 comprises as well the multiple objective formulation  $R(x, \delta) = (E(x, \delta), \text{Var}[F(x, \delta, p)])$ , where the expected value is optimized while the variance is minimized. This is probably the most frequent approach adopted by analysts.

When the DM are able to express some criteria about the output (case 1), it is possible define more specific problems. For instance, the variance does not necessarily have to be minimized but simply bounded inside a threshold of acceptance. Other constraints are possible. One example of this in evolutionary computation is constituted by Deb's multiple objective robust definitions 2 and 4 (Deb & Gupta, 2005), where the percentage of deviation between the single and expected values of  $F(x)$  is constrained a priori.

- Optimization of the worst case: this is a less common but still valid criterion used in case 2, and corresponds to min-max or max-min problems. Sometimes it is considered as a conservative criterion, but its usefulness depends on the problem. This criterion is often employed in combinatorial problems. A well-known group of robustness criteria that consider worst-case (*in discrete domain*) are Kouvelis and Yu's (1997) metrics.

*Approach 2: effective domain assessment*

When it is not possible to retrieve neither information nor suitable assumptions about  $\delta$ , analysts may try to assess what is the effective domain within which the system remains valid. In robust design, the constraint satisfaction problem is a typical example of what we are talking about.

Now consider case 3 where the DM can state some goals or constraints on  $F(x)$ . If  $F_{\min}$  and  $F_{\max}$  can be identify, then a valid approach consists of identifying that value of  $x$  that allows the maximal deviation without missing the requirements  $G_j(x, \delta, p) \leq b_j$  and  $F_{\min} \leq F(x, \delta, p) \leq F_{\max}$ . Therefore robustness criterion to be maximized is  $R(x, \delta) = \text{dist}(\delta_{\min}, \delta_{\max})$ . Such distance could be defined in different ways (Milanese et al., 1996). For example in (Rocco et al., 2003; Rocco, 2005; Salazar & Rocco) the authors use the Maximal volumen Inner Box (MIB) distance formulated as:

$$\text{dist}(\delta_{\min}, \delta_{\max}) = \prod_i |\delta_{\min, i} - \delta_{\max, i}|$$

where  $\delta_{*, i}$  is the  $i$ th component of vector  $\delta_*$ , and is applied to single and multiple objective robustness problems. Then, stage 2 is carried out with Interval Arithmetic and Evolutionary Computation.

### Approach 3: minimal information approach

Case 4 is the hardest situation that an analyst could cope with. It is characterized by an inability of describing  $\delta$  plus ignorance about the range of function  $F(x)$ . The consequence is that the preceding approaches cannot be employed successfully. To our best knowledge this kind of situations has not been studied before, perhaps because, even when this circumstance might arise -and in fact it does- in real problems, it is far away from being frequent. Nonetheless, a methodology based on the minimal information that the DM can articulate was introduced by Salazar et al. (2006), when dealing with a particular flow-shop scheduling problem.

The idea is to assume plausible values of  $\delta_{\min}$ ,  $\delta_{\max}$  and a uniform PDF, in order to apply the uncertainty propagation approach just to figure out the zone of optimality in the objective space, in such a way that the DM can have a better panorama of the behaviour of  $F(x)$ , as well as obtaining some optimal solutions. Afterwards, since there's no reliable information to suitably describe the uncertainty, the original assumption about  $\delta$  is discarded and approach 2 is applied, fixing  $F_{\min}$  and  $F_{\max}$  in accordance with DM's preferences. Given that an increment in the range of  $\delta$  could reduce the level of optimality of the previously found solutions, the condition to be imposed is to allow any displacement interval considered indifferent by the DM. The solution with the MIB is the chosen one.

### 2<sup>nd</sup> Stage: solving the problem

Finally, once the robustness criteria are correctly identified, the 2<sup>nd</sup> stage consists in determining a suitable methodology to find the solution. This is actually an open field characterized by recent innovations and contributions [Jin] [Paenke] [Ong]. However, it is important to remark that all the contributions in this area are subject to a particular concept from those mentioned earlier on. If the concept is no longer applicable in a particular problem, the strategies developed to accomplish the 2<sup>nd</sup> stage must be tailored in the best case. Thus we have two areas of research, the conceptual one and the implementation one.

### 3. Final comments

Even when the classification of the different concepts of robustness is in any case not a new idea

(see e.g. the *two families* of approaches in Aloulou et al., 2005), we believe that the perspective presented here is useful to clarify where and how the different contributions made in this area fit and relate to each other. Likewise it allows identifying the difficulties and the actual limitations for solving the 2<sup>nd</sup> stage. Moreover, the generic formulation proposed is useful to understand why there are so many variants of the same concept and, at the same time, it's a nice way of joining all them together. As we mentioned in the introduction, this information-based perspective must be extended to consider other sort of systems and their derived problems.

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