

# Bifurcation analysis in a model of urban environmental quality

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## Abstract

In this paper a bifurcation analysis of some special cases of an urban quality model proposed in [2] is performed. The bifurcation parameters are the population growth rate and the taxes needed to prevent bad quality to be predominant in the urban environment. It is shown that a generalized saddle-node bifurcation happens for reasonable ranges of the parameter values and a interpretation of this phenomenon in urban and economic terms is also offered.

**KEY WORDS:** Bifurcation, ODE Model, Quality, Environmental Quality, Urban Quality.

## 1 Introduction

Studies on urban development and sustainability are becoming more and more useful as the growth of urban areas expands extensively in many parts of the world [1, 5, 6, 8, 9]. Mathematics is one of the most useful and interesting tools in the analysis of these new type of problems, and models are a widespread way of thinking about them.

The mathematical modelling of urban environmental quality is a result of considering the interaction between three main agents: A) a measure of quality, B) population, and C) a measure of *bad* quality. In this paper we study the evolution of a urban settlement *ex novo*. Quality is usually measured as urban equipment in  $m^2$  per housing unit, and a suitable quantity is prescribed by local

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or national regulations and laws. Population is supposed in a first instance to have a logistic growth. For a start, its growth is considered to be in path with that of quality. Bad quality, representing not the lack of equipment but rather the amount of existing and spoiled equipment, is also measured in  $m^2$  per housing unit and its evolution is dictated by the interaction between population, quality and economic considerations.

We do not take into account spatial distribution effects (for a view of these problems, see [4]), so an ODE model is proposed (see [2] for extensive details on the modelling process):

$$\begin{aligned}x' &= r_1 \left( \frac{1}{1 + My} \right) x \left( 1 + \frac{x}{K} \right) - Dxy \\y' &= r_2 y \left( 1 - \frac{y}{C} \right) - Gy(x - z) \\z' &= D^*xy - Hy \left( \frac{z}{P + z} \right)\end{aligned}$$

where  $x = x(t)$ ,  $y = y(t)$ ,  $z = z(t)$  stand respectively for quality, population and bad quality. The linear growth rate  $r_1$  for quality is modulated by the expression  $\frac{1}{1 + My}$  having an *educational* meaning.  $K$  is the quality standard fixed by law, and the interaction term  $-Dxy$  is interpreted as a *destructive* term due to the use of equipment by population.  $r_2$  is the linear growth rate of population,  $C$  is its carrying capacity, while the expression  $-Gy(x - z)$  represents the *cost of maintaining* quality.  $D^*$  stands for the fraction of destroyed quality that turns into bad quality, and the last term  $-Hy \left( \frac{z}{P + z} \right)$  indicates that even bad quality may have a limit.

The above system depends on ten parameters, but it can be written in dimensionless form by some judicious choices of units to make it easier to manipulate. For a good account on these techniques, see [3]. Hopefully, the number of parameters will be reduced. This is in fact what happens. Let:

$$x = Kx^*, \quad y = Cy^*, \quad t = \frac{t^*}{r_1}, \quad \text{and} \quad z = Cz^*$$

to find –after some algebra and dropping the asterisks– the simpler system in these new non-dimensional variables whose ranges are the closed interval  $[0, 1]$ :

$$\begin{aligned}x' &= \frac{x}{1 + \beta y} (1 - x) - \gamma xy \\y' &= \alpha y (1 - y) - \delta xy \\z' &= \gamma^* xy - \eta \frac{yz}{\phi + z}\end{aligned}$$

where the *only* seven dimensionless parameters have the following meaning:  $\alpha = \frac{r_2}{r_1}$  is the relative growth rate of population versus quality,  $\beta = MC$  is the educational parameter,  $\gamma = \frac{CD}{r_1}$  is the quality destruction rate,  $\delta = \frac{GK}{r_1}$  measures the effect of taxes,  $\eta = \frac{HC}{Kr_1}$  indicates that some bad quality can be considered as usable for certain population,  $\phi = PK$  is a half saturation constant, and  $\gamma^* = \frac{CD^*}{r_1}$  is obviously the fraction of destroyed quality that turns into unusable or bad quality.

## 2 Singular points

The system we shall consider is a nonlinear one, and to grasp its main features we start by computing its singular points and their linear stability. Solving the algebraic system

$$\begin{aligned} 0 &= \frac{x}{1 + \beta y} (1 - x) - \gamma xy \\ 0 &= \alpha y (1 - y) - \delta xy \\ 0 &= \gamma^* xy - \eta \frac{yz}{\phi + z} \end{aligned}$$

yields the coordinates of the singular points. It is plain to see that the  $z$ -axis is a set of unstable singular points: Indeed, any  $(0, 0, z)$  will annulate the three equations. Every point in the straight line  $\{x = 1, y = 0\}$  is also a singular one, and the dynamics in the plane  $y = 0$  is restricted to movements along lines parallel to the  $x$ -axis and towards the line  $\{x = 1, y = 0\}$ , mimicking the logistic dynamics on the  $x$ -axis.

Therefore, factoring out  $x$  and  $y$  in the appropriate equations the above system is equivalent to this one:

$$\begin{aligned} x + \gamma y + \gamma \beta y^2 &= 1 \\ \delta x + \alpha y - \delta z &= \alpha \\ \gamma^* \phi x + \gamma^* xz - \eta z &= 0 \end{aligned}$$

From a geometric viewpoint, the first and third equations represent two cylinders, respectively parallel to the  $z$ -axis and to the  $y$ -axis. These surfaces hopefully will determine a skew curve whose intersections with the plane represented by the second equation will be the singular points. The following analysis shows what is actually happening.

Let us write the three above equations in parametric form, taking  $y$  and  $z$  as

parameters:

$$\begin{aligned}x_1(y, z) &= 0, \text{ here } x_1(y) = 0 \\x_2(y, z) &= 0 \\x_3(y, z) &= 0, \text{ here } x_3(z) = 0\end{aligned}$$

to obtain the following representation

$$\begin{aligned}x &= 1 - \gamma(y + \beta y^2) \\x &= \frac{\alpha}{\delta}(1 - y) + z \\x &= \frac{\eta}{\gamma^*} \frac{z}{\phi + z} \\(y, z) &\in [0, 1] \times [0, 1]\end{aligned}$$

For the first equation in this last set,  $\frac{dx}{dy} = -(\gamma + 2\beta y) < 0$  for any values of  $\beta$  and  $\gamma$  with  $y > 0$ , so the parabola represented by this equation in the  $(x, y)$ -plane is always decreasing and has the positive intersection  $y = \frac{-\gamma + \sqrt{\gamma^2 + 4\beta\gamma}}{2\beta\gamma}$  with the  $y$ -axis. Similarly, the third equation yields  $\frac{dx}{dz} = \frac{\eta\phi}{\gamma^*} \frac{1}{(\phi + z)^2} > 0$  for every  $z$  so the cylinders represented by both equations do intersect along a curve, and a stretch thereof is contained in the cube  $[0, 1]^3$  for reasonable values of the parameters. Now, the plane  $x = \frac{\alpha}{\delta}(1 - y) + z$  may have or have not points in common with this curve, and this depends only on the value taken by the combination  $\frac{\alpha}{\delta}$ . See Figure 1.

The parameter combinations  $\frac{\alpha}{\delta}$  and  $\frac{\eta}{\gamma^*}$  are the governing values in the second and third equations respectively.  $\phi$  should take any value less than 1,  $\eta$  and  $\gamma^*$  must be roughly of the same order and  $\frac{\eta}{\gamma^*}$  can be set to 1.

### 3 A bifurcation analysis

A bifurcation is a phenomenon by which a variation on some (set of) parameter values involves a qualitative change of the phase portrait of the system [10]. For instance, the onset of oscillative patterns in many model is often a result of a Hopf bifurcation, see an example in [7].

In our model  $\phi$  measures how fast the interaction between population and bad quality becomes a linear function of population: The smaller  $\phi$  is, the faster

Figure 1: COMPUTATION OF SINGULAR POINTS. IN THIS CASE TWO SINGULAR POINTS APPEAR FOR THE CHOICE OF VALUES  $\gamma = 1$ ,  $\beta = 0.5$ ,  $\eta/\gamma^* = 1$ ,  $\phi = 1$ ,  $\alpha/\delta = 0.1$ .

it happens. Therefore we shall consider  $\phi$  a most interesting parameter. On the other hand,  $\frac{\alpha}{\delta}$  will be the another important parameter.

By gridding the plane of the variables  $\phi$  and  $\frac{\alpha}{\delta}$  and computing the number of solutions of the system for every pair  $(\phi, \frac{\alpha}{\delta})$  –with fixed values of the other five parameters– a bifurcation diagram like the one shown in Figure 2 can be built. The construction can be programmed in Mathematica or Maple as a curve fitting exercise.

BIFURCATION DIAGRAM IN THE PLANE SPANNED BY  $\phi$  AND  $\alpha/\delta$ . THE OTHER PARAMETERS ARE THE SAME AS IN FIGURE 1.

The diagram shows that there is a region in parameter plane below the *bifur-*

*cation curve* for which no singular points exist. When the pair  $(\phi, \frac{\alpha}{\delta})$  crosses the bifurcation curve, two singular points suddenly appear “out of nothing”. One of them is a stable node, as is shown by computing the sign of the Jacobian matrix eigenvalues: All of them are negative according to the Routh-Hurwitz criterion applied to the Jacobian matrix. The other one has a positive eigenvalue and is thus unstable.

For instance, when  $(\phi, \frac{\alpha}{\delta}) = (0.1, 0.6)$  with  $\alpha = 0.1$ , the two singular points are  $(0.33, 0.52, 0.05)$ , an unstable saddle point, and the stable node  $(0.73, 0.24, .27)$ . If we let  $\frac{\alpha}{\delta}$  grow and maintain  $\phi = 0.1$ , both points coalesce at  $(0.55, 0.376, 0.123)$  for the bifurcation value  $\frac{\alpha}{\delta} = 0.688205$  and then disappear. As a rule, the  $x$ -coordinate of the stable point decreases with growing  $\frac{\alpha}{\delta}$ , while for the unstable point the opposite is true.

We conclude that we have found a *generalized saddle-node* bifurcation [10] as the main feature of the model behaviour. The unstable point has a two-dimensional stable manifold, such that if the initial conditions happen to be in it, the system will eventually move towards the unstable point, otherwise the stable node is asymptotically approached.

## 4 Discussion and further comments

The results obtained in the above section have an interesting explanation in terms of the leading parameters. Let us make an interpretation of Figure 2.

The half-saturation constant  $\phi$  tells us how fast the quotient  $\frac{z}{\phi + z}$  approaches a constant value: The smaller  $\phi$  is, the faster the quotient looks constant. Therefore, a small  $\phi$  indicates that elimination of bad quality is achieved quite soon by a constant effort  $\eta$  of the population. Moreover, for small  $\phi$  there is a wide range of  $\frac{\alpha}{\delta}$  values for which there are feasible singular points. Now  $\frac{\alpha}{\delta}$  can grow either by taking larger  $\alpha$  or smaller  $\delta$ . Large values of  $\alpha$  indicate a rapid population settlement, while small  $\delta$ 's are a signal of cheap maintenance taxes. Some compromise should be made between the parameter values in order to obtain an optimal –in a sense to be specified– behaviour: A certain quality must be kept, as well as a reasonable population number and a not too large or “sustainable” bad quality.

If  $\phi$  is large, say  $\phi > 0.4$ , then feasible singular points do exist only if  $\frac{\alpha}{\delta}$  is rather small: A successful fight against bad quality needs either a small population growth or (maybe “and”) large maintenance taxes.

A complete bifurcation analysis of the proposed model is out of the scope of this paper and it will be undertaken elsewhere. Nevertheless, the method is a valid one: First, choose a set of reasonable values for all parameters; second, make

of choice of interesting combinations of some of them (in the above study, only two were considered); and third, perform an analysis of singular points and their stability depending on these combinations. Even for very simple models, closed form formulas will be rarely found, and a numerical procedure shall be devised for each case.

A thorough future analysis will deal with the possibility of cyclic and chaotic behaviour in the development of urban settlements and their qualitative conditions depending on definite regions of parameter space.

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