Questions:

(1) What is the range of $g(n)$? For example, it appears that neither 14 nor 24 is in the range.
(2) What is the asymptotic behavior of $g(n)$? Is $g(n)$ asymptotic to $c\log_2(F_n)$ for some constant c as n approaches infinity?
(3) What is the limit of $g(n)/n$ as n approaches infinity? Does it equal $c\log_2(\alpha)$?
(4) Can we determine an explicit formula for $g(n)$?

Solution 2 by Albert Stadler, Herrliberg, Switzerland.

We use a theorem by Pillai [1] stating that if $R(n)$ is the least integer r for which the rth iterate of the Euler totient function evaluated at n equals 1, then

$$R(n) \le \left\lfloor \frac{\log n}{\log 2} \right\rfloor + 1.$$

Clearly, $R(F_1) = R(1) = 1$. If $n \ge 2$, then by Pillai's theorem and that $F_n \le 2^{n-1}$,

$$R(F_n) \le \left\lfloor \frac{\log F_n}{\log 2} \right\rfloor + 1 \le \left\lfloor \frac{\log 2^{n-1}}{\log 2} \right\rfloor + 1 = n,$$

and we are done.

Editor's Note: Fedak observed that if $\frac{m}{2^{n-1}} \le 2$, then $\phi^n(m) = 1$. This leads to the desired result immediately because $F_n < 2^n$. Park proved that if $\phi^{k-2}(a) \ge 3$, then $\phi^k(a) \le \frac{1}{2^{k-1}}\, a$, which completes the proof because $F_n \le 2^{n-1}$.

References

[1] S. S. Pillai, *On a function connected with* $\varphi(n)$, Bull. Amer. Math. Soc., **35** (1929), 837–841.

Also solved by Saunak Bhattacharjee (undergraduate), I. V. Fedak, Heng Gao, Ho Park, Raphael Schumacher (graduate student), and David Terr.

A Consequence of the Minkowski Inequality

B-1281 Proposed by Ivan V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine. (Vol. 59.1, February 2021)

For all positive integers n and m, prove that

$$L_1 + \sqrt[n]{\sum_{k=1}^{m} L_k^n} \le \sqrt[n]{\sum_{k=1}^{m} F_{k+1}^n} + F_{m+1}.$$

Solution by Ángel Plaza, Universidad de Las Palmos de Gran Canaria, Gran Canaria, Spain.

Because $L_1 = 1$, $L_k = F_{k-1} + F_{k+1}$, and $\sum_{k=1}^{m-1} F_k = F_{m+1} - 1$, the proposed inequality may be written equivalently as

$$\sqrt[n]{\sum_{k=1}^{m}\left(F_{k-1}+F_{k+1}\right)^{n}} \leq \sqrt[n]{\sum_{k=1}^{m} F_{k+1}^{n}}+\sum_{k=1}^{m} F_{k-1}.$$

Now, because $\sqrt[n]{\sum_{k=1}^{m} F_{k-1}^{n}} \leq \sum_{k=1}^{m} F_{k-1}$, it is enough to prove that

$$\sqrt[n]{\sum_{k=1}^{m}\left(F_{k-1}+F_{k+1}\right)^{n}} \leq \sqrt[n]{\sum_{k=1}^{m} F_{k+1}^{n}}+\sqrt[n]{\sum_{k=1}^{m} F_{k-1}^{n}},$$

which is the Minkowski inequality.

Also solved by Dmitry Fleischman, Haydn Gwyn (undergraduate), Hideyuki Ohtsuka, Albert Stadler, and the proposer.

Both Summations Are Telescopic

B-1282 Proposed by Hideyuki Ohtsuka, Saitama, Japan. (Vol. 59.1, February 2021)

For any positive integer n, find closed form expressions for the sums

$$\sum_{k=1}^{n} F_{3k}F_{3k+1}, \qquad \text{and} \qquad \sum_{k=1}^{n} F_{2F_{3k}}F_{2F_{3k+1}}.$$

Solution by Raphael Schumacher (graduate student), ETH Zurich, Switzerland.

We have for all $k \in \mathbb{N}$ the identities

$$\frac{F_{3k+2}^{2}-F_{3k-1}^{2}}{4}=\frac{\left(F_{3k+1}+F_{3k}\right)^{2}-\left(F_{3k+1}-F_{3k}\right)^{2}}{4}=F_{3k}F_{3k+1},$$

$$F_{2F_{3k}}F_{2F_{3k+1}}=F_{F_{3k+1}+F_{3k}}^{2}-F_{F_{3k+1}-F_{3k}}^{2}=F_{F_{3k+2}}^{2}-F_{F_{3k-1}}^{2}.$$

In the second identity, we have used for $a \geq b$ the identity

$$F_{2a}F_{2b}=F_{a+b}^{2}-F_{a-b}^{2},$$

which can be derived from the Binet's formula, the Catalan's identity, or the product formula $5F_sF_t = L_{s+t}-(-1)^tL_{s-t}$. By telescoping, it follows from the above two identities with $F_2 = 1$ that

$$\sum_{k=1}^{n} F_{3k}F_{3k+1}=\sum_{k=1}^{n} \frac{F_{3k+2}^{2}-F_{3k-1}^{2}}{4}=\frac{F_{3n+2}^{2}-1}{4},$$

$$\sum_{k=1}^{n} F_{2F_{3k}}F_{2F_{3k+1}}=\sum_{k=1}^{n}\left(F_{F_{3k+2}}^{2}-F_{F_{3k-1}}^{2}\right)=F_{F_{3n+2}}^{2}-1.$$

Editor's Note: Several solvers used the Binet's formulas to set up a summation of arithmetic progressions in α and β, thereby obtaining $\sum_{k=1}^{n} F_{3k}F_{3k+1}=\frac{1}{20}\left(L_{6n+4}-5-2(-1)^{n}\right)$. Gwyn, using a slightly different set up, showed that $\sum_{k=1}^{n} F_{3k}F_{3k+1}=\frac{1}{4}\left(2F_{3n+2}F_{3n+3}-F_{6n+4}-1\right)$. Meanwhile, many solvers used the identity $F_{2F_{3k}}F_{2F_{3k+1}}=\frac{1}{5}\left(L_{2F_{3k+2}}-L_{2F_{3k-1}}\right)$ to derive the result $\sum_{k=1}^{n} F_{2F_{3k}}F_{2F_{3k+1}}=\frac{1}{5}\left(L_{2F_{3n+2}}-3\right)$.