Hence,

$$\lim_{n\to\infty}\left(\lim_{x\to\infty}\left((f(x+1))^{\frac{L_n}{(x+1)L_{n+1}}}-(f(x))^{\frac{L_n}{xL_{n+1}}}\right)x^{\frac{L_{n-1}}{L_{n+1}}}\right)$$
$$=\lim_{n\to\infty}\lim_{x\to\infty}\left(f(x+1)^{\frac{u_n}{x+1}}-f(x)^{\frac{u_n}{x}}\right)x^{1-u_n}=\lim_{n\to\infty}\lim_{x\to\infty}f(x)^{\frac{u_n}{x}}(v_n(x)-1)x^{1-u_n}$$
$$=\lim_{n\to\infty}\lim_{x\to\infty}f(x)^{\frac{u_n}{x}}\left(\frac{v_n(x)-1}{\ln v_n(x)}\right)x^{1-u_n}\ln v_n(x)$$
$$=\lim_{n\to\infty}\left(\lim_{x\to\infty}\left(\frac{f(x)^{\frac{1}{x}}}{x}\right)^{u_n}\cdot\lim_{x\to\infty}\left(\frac{v_n(x)-1}{\ln v_n(x)}\right)\cdot\lim_{x\to\infty}\ln(v_n(x)^x)\right)$$
$$=\lim_{n\to\infty}\left(\frac{a}{e}\right)^{u_n}\cdot 1\cdot\ln e^{u_n}=\frac{1}{\alpha}\left(\frac{a}{e}\right)^{\frac{1}{\alpha}}.$$

Also solved by Dmitry Fleischman and Albert Stadler.

The generating function of the product of the Fibonacci and Tribonacci numbers

H-855 Proposed by Robert Frontczak, Stuttgart, Germany (Vol. 58, No. 2, May 2020)

Let $(T_n)_{n\geq 0}$ be the sequence of Tribonacci numbers given by $T_0=0$, $T_1=T_2=1$, and $T_n=T_{n-1}+T_{n-2}+T_{n-3}$ for $n\geq 3$. Define the functions

$$G_{FT}(z)=\sum_{n=0}^{\infty}F_nT_nz^n \quad\text{and}\quad G_{LT}(z)=\sum_{n=0}^{\infty}L_nT_nz^n.$$

Show that for $k\geq 1$, we have

$$G_{FT}(2^{-2k})=\frac{2^{4k}(2^{6k}-2^{2k}-1)}{2^{12k}-2^{10k}-2^{8k+2}-2^{6k+2}-2^{6k}-2^{4k+1}+2^{2k}-1}$$

and

$$G_{LT}(2^{-2k})=\frac{2^{4k}(2^{6k}+2^{4k+1}+2^{2k}+3)}{2^{12k}-2^{10k}-2^{8k+2}-2^{6k+2}-2^{6k}-2^{4k+1}+2^{2k}-1}.$$

Solution by Ángel Plaza, Gran Canaria, Spain

Because the generating function for the Tribonacci numbers is

$$T(z)=\sum_{n=0}^{\infty}T_nz^n=\frac{z}{1-z-z^2-z^3},$$

it follows by the Binet's formulas that

$$G_{FT}(z)=\frac{T(\alpha z)-T(\beta z)}{\sqrt{5}}\quad\text{and}\quad G_{LT}(z)=T(\alpha z)+T(\beta z)$$

from where

$$\begin{aligned} G_{FT}(z) &= \frac{z\left(z^3+z^2-1\right)}{z^6-z^5+2z^4+5z^3+4z^2+z-1} \quad \text{and} \\ G_{LT}(z) &= \frac{z\left(3z^3+z^2+2z+1\right)}{z^6-z^5+2z^4+5z^3+4z^2+z-1}, \quad \text{so} \\ G_{FT}(2^{-2k}) &= \frac{2^{4k}(2^{6k}-2^{2k}-1)}{2^{12k}-2^{10k}-2^{8k+2}-2^{6k+2}-2^{6k}+2^{2k}-1} \quad \text{and} \\ G_{LT}(2^{-2k}) &= \frac{2^{4k}(2^{6k}+2^{4k+1}+2^{2k}+3)}{2^{12k}-2^{10k}-2^{8k+2}-2^{6k+2}-2^{6k}+2^{2k}-1}. \end{aligned}$$

Also solved by Brian Bradie, Dmitry Fleischman, Kapil Kumar Gurjar, Raphael Schumacher, Jason L. Smith, Albert Stadler, and the proposer.

The sum of a series with Fibonacci and triangular numbers

H-856 Proposed by Robert Frontczak, Stuttgart, Germany (Vol. 58, No. 2, May 2020)

Let T_n denote the nth triangular number; i.e., $T_n = n(n+1)/2$. Show that

$$\sum_{n=0}^{\infty} T_n \cdot \frac{F_n}{2^{n+2}} = F_7 \quad \text{and} \quad \sum_{n=0}^{\infty} T_n \cdot \frac{L_n}{2^{n+2}} = L_7.$$

Solution by Hideyuki Ohtsuka, Saitama, Japan

By the identity

$$\frac{d^2}{dx^2}\sum_{n=0}^{\infty} x^{n+1} = \frac{d^2}{dx^2}\frac{x}{1-x} \quad \text{for} \quad |x|<1,$$

we have

$$\sum_{n=0}^{\infty} n(n+1)x^{n-1} = \frac{2}{(1-x)^3}.$$

Therefore, we have

$$\sum_{n=0}^{\infty} T_n \frac{x^n}{4} = \frac{x}{4(1-x)^3}.$$

Evaluating at $x=\alpha/2$ and $x=\beta/2$ the above identity we get

$$\sum_{n=0}^{\infty} T_n \cdot \frac{\alpha^n}{2^{n+2}} = \frac{29+13\sqrt{5}}{2} \quad \text{and} \quad \sum_{n=0}^{\infty} T_n \cdot \frac{\beta^n}{2^{n+2}} = \frac{29-13\sqrt{5}}{2}.$$

We obtain the desired identities by using the Binet formulas for F_n and L_n.

Also solved by Michel Bataille, Brian Bradie, Charles K. Cook, Dmitry Fleischman, Kapil Kumar Gurjar, Ángel Plaza, Diego Rattaggi, Henri Ricardo, Raphael Schumacher, J. N. Senadheere, Jason L. Smith, Albert Stadler, David Terr, and the proposer.

Note: The reader is referred to the retraction notice from the end of the Advanced Problem Section of **Vol. 58, No. 3, August 2020** issue.