and the coefficient of x^n in $x^2G_F(x)$ is F_{n-2}. Thus,

$$F_{n-2} = \sum_{i\geq 1} \sum_{s_1+s_2+\cdots+s_i=n} T_{s_1-1}T_{s_2-1}\cdots T_{s_i-1}.$$

Finally, we note that the sum on the right side is zero when $i > n$ (because we cannot have more than n positive integers sum to n), and when $i = n$ (because in this case each $s_k = 1$ and the summand is simply the product of n copies of $T_0 = 0$. Hence, we conclude that

$$F_{n-2} = \sum_{i=1}^{n-1} \sum_{s_1+s_2+\cdots+s_i=n} T_{s_1-1}T_{s_2-1}\cdots T_{s_i-1}.$$

as required.

Reference

[1] R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics*, 2nd ed., Addison-Wesley Publishing Company, Reading, MA, 1994.

Also solved by Dmitry Fleischman, Raphael Schumacher, Albert Stadler, and the proposer.

A perfect square

H-858 Proposed by Muneer Jebreel Karama, Hebron, Palestine (Vol. 58, No. 3, August 2020)

Show that

$$\frac{1}{2}\left((2F_nF_{n+1})^8 + (F_{n-1}F_{n+2})^8 + F_{2n+1}^8\right)$$

is a perfect square for all $n \geq 0$.

Solution by Hideyuki Ohtsuka, Saitama, Japan

The desired expression is a square because

$$F_{n-1}F_{n+2} = F_{n+1}^2 - F_n^2, \quad F_{2n+1} = F_n^2 + F_{n+1}^2 \quad \text{(see [1](11),(12))}$$

and

$$\frac{1}{2}((2ab)^8 + (a^2-b^2)^8 + (a^2+b^2)^8) = (a^8 + 14a^4b^4 + b^8)^2.$$

Reference

[1] S. Vajda, *Fibonacci and Lucas Numbers and the Golden Section*, Dover, 2008.

Also solved by Brian Bradie, Dmitry Fleischman, Wei-Kai Lai, Ángel Plaza, Raphael Schumacher, Jason L. Smith, Albert Stadler, and the proposer.

A series with Fibonacci numbers and values of the Riemann zeta function

H-859 Proposed by Robert Frontczak, Stuttgart, Germany (Vol. 58, No. 3, August 2020)

Prove that

$$\sum_{n\geq 1} \zeta(2n+1)\frac{F_{2n}}{5^n} = \frac{1}{2},$$

where $\zeta(k) = \sum_{n\geq 1} 1/n^k$ for $k \geq 2$ is the Riemann zeta function.