H-881 Proposed by Hideyuki Ohtsuka, Saitama, Japan

For any positive integers r and n, prove that

$$\sum_{k=0}^{n}\binom{2n}{n-k}\frac{F_{4rk}}{F_{4r}}=\sum_{k=0}^{n-1}\binom{2k}{k}L_{2r}^{2n-2k-2}.$$

H-882 Proposed by Robert Frontczak, Stuttgart, Germany

Prove the following identities for the Fibonacci and Lucas numbers

$$\sum_{k=0}^{n}\binom{n}{k}\frac{F_k+L_k}{k+1}=\frac{F_{2n+1}+L_{2n+1}}{n+1}\quad\text{and}\quad\sum_{k=0}^{n}\binom{n}{k}\frac{F_k+L_k}{(k+1)(k+2)}=\frac{F_{2n+2}+L_{2n+2}-2}{(n+1)(n+2)}.$$

Errata. In the right–hand side of **H-868**, the term to be summed inside the most inner sum should be

$$\frac{k^{2\ell}(2^{2\ell}-1)}{\ell d^{2\ell}}\zeta(2\ell)\quad\text{instead of}\quad\frac{k^{2\ell}(2^{2\ell}-1)}{\ell 2^{2\ell}}\zeta(2\ell).$$

SOLUTIONS

Identities with generalized-balancing numbers

H-844 Proposed by Robert Frontczak, Stuttgart, Germany (Vol. 57, No. 3, August 2019)

Let $B_n=B_n(\alpha,\beta)$ be a generalized balancing number given by $B_0(\alpha,\beta)=\alpha,\ B_1(\alpha,\beta)=\beta$, and for $n\geq 2$,

$$B_n(\alpha,\beta)=6B_{n-1}(\alpha,\beta)-B_{n-2}(\alpha,\beta).$$

Prove that

$$\sum_{k=0}^{2n}\binom{4n}{2k}B_{2k}(\alpha,\beta)=(2^{6n-1}+2^{4n-1})B_{2n}(\alpha,\beta)$$

and

$$\sum_{k=0}^{\lfloor(4n-1)/2\rfloor}\binom{4n}{2k+1}B_{2k}(\alpha,\beta)=(2^{6n-1}-2^{4n-1})B_{2n}(\alpha,\beta).$$

Solution by Ángel Plaza, Gran Canaria, Spain

We will use the following identities for the generalized balancing numbers

$$\sum_{k=0}^{m}\binom{2m}{2k}B_{2k}(\alpha,\beta)=(2^{3m-1}+2^{2m-1})B_m(\alpha,\beta)$$

and

$$\sum_{k=0}^{\lfloor(2m-1)/2\rfloor}\binom{2m}{2k+1}B_{2k+1}(\alpha,\beta)=(2^{3m-1}-2^{2m-1})B_m(\alpha,\beta),$$

which are respectively identities (2.16) and (2.17) in Proposition 2.1 in [1]. By letting $m=2n$, the proposed identities follow.

Reference

[1] R. Frontczak, *Identities for generalized balancing numbers*, Notes on Number Theory and Discrete Mathematics, **25** (2019), 169–180.

Also solved by Brian Bradie, David Terr, and the proposer.

Some double limits with ratios of Lucas numbers

H-845 Proposed by D. M. Bătineţu, Bucharest, Romania, and N. Stanciu, Buzău, Romania (Vol. 57, No. 3, August 2019)

Compute

$$\lim_{n\to\infty}\left(\lim_{x\to\infty}\left((f(x+1))^{L_n/((x+1)l_{n+1})}-(f(x))^{L_n/(xL_{n+1})}\right)x^{L_{n-1}/L_{n+1}}\right),$$

where $f:\mathbb{R}_+^*\to\mathbb{R}_+^*$ is a function that satisfies $\lim_{x\to\infty} f(x+1)/(xf(x))=a\in\mathbb{R}_+^*$.

Solution by the proposers

We denote $u_n=L_n/L_{n+1}$. We have

$$\lim_{n\to\infty}u_n=\lim_{n\to\infty}\frac{\alpha^n+\beta^n}{\alpha^{n+1}+\beta^{n+1}}=\frac{1}{\alpha}.$$

We also have

$$\begin{aligned}\lim_{x\to\infty}\frac{(f(x))^{\frac{1}{x}}}{x} &= \lim_{n\to\infty}\frac{f(n)^{\frac{1}{n}}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{f(n)}{n^n}}\\ &= \lim_{n\to\infty}\frac{f(n+1)}{(n+1)^{n+1}}\cdot\frac{n^n}{f(n)}=\lim_{n\to\infty}\frac{f(n+1)}{nf(n)}\left(\frac{n}{n+1}\right)^{n+1}=\frac{a}{e}.\end{aligned}$$

We denote

$$v(x)=\frac{f(x+1)^{\frac{L_n}{(x+1)L_{n+1}}}}{f(x)^{\frac{L_n}{xL_{n+1}}}}=\left(\frac{f(x+1)^{\frac{1}{x+1}}}{f(x)^{\frac{1}{x}}}\right)^{u_n}.$$

We have $\lim_{x\to\infty}v(x)=1$, so

$$\lim_{x\to\infty}\frac{v(x)-1}{\ln v(x)}=1$$

and

$$\lim_{x\to\infty}(v(x))^x=\lim_{x\to\infty}\left(\frac{f(x+1)}{f(x)}\cdot\frac{1}{f(x+1)^{\frac{1}{x+1}}}\right)^{u_n}=\lim_{x\to\infty}\left(\frac{a(x+1)}{(f(x+1))^{\frac{1}{x+1}}}\right)^{u_n}=e^{u_n},$$

therefore

$$\lim_{n\to\infty}\left(\lim_{x\to\infty}(v(x))^x\right)=e^{\frac{1}{\alpha}}.$$