References

[1] T. Agoh, *Convolution identities for Bernoulli and Genocchi polynomials*, The Electronic Journal of Combinatorics, **21.1** (2014), #P1.65.
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Also solved by Brian Bradie, Dmitry Fleischman, G. C. Greubel, Raphael Schumacher, and Albert Stadler.

Lower bounds for some sums involving Lucas numbers

H-853 Proposed by Ángel Plaza and Sergio Falcón, Gran Canaria, Spain (Vol. 58, No. 1, February 2020)

Let L_n be the nth k-Lucas number given by the recurrence $L_{n+2} = kL_{n+1} + L_n$ for all $n \geq 0$, with $L_0 = 2$, $L_1 = k$. Prove that

$$\text{(i)} \sum_{j=1}^{n} \frac{L_j^2}{\sqrt{L_j + 1}} \geq \frac{(L_n + L_{n+1} - k - 2)^2}{k\sqrt{kn(L_n + L_{n+1} + k(n-1) - 2)}};$$

$$\text{(ii)} \sum_{j=1}^{n} \frac{L_j^4}{\sqrt{L_j^2 + 1}} \geq \frac{(L_{2n+1} + k((-1)^n - 2))^2}{k\sqrt{kn(L_{2n+1} + k(n - 2 + (-1)^n))}}.$$

Solution by the proposers

The inequalities follow by Jensen's inequality. Note that the function $f(x) = \dfrac{x^2}{\sqrt{x+1}}$ is convex because $f''(x) = \dfrac{3x^2 + 8x + 8}{4(x+1)^{5/2}} > 0$. Therefore,

$$\begin{aligned}
\sum_{j=1}^{n} \frac{L_j^2}{\sqrt{L_j + 1}} &\geq n \cdot \frac{\left(\frac{\sum L_j}{n}\right)^2}{\sqrt{\frac{\sum L_j}{n} + 1}} \\
&= n \cdot \frac{\left(\frac{L_n + L_{n+1} - k - 2}{kn}\right)^2}{\sqrt{\frac{L_n + L_{n+1} - k - 2}{kn} + 1}} = \frac{(L_n + L_{n+1} - k - 2)^2}{k\sqrt{kn(L_n + L_{n+1} + k(n-1) - 2)}},
\end{aligned}$$

where we use $\displaystyle\sum_{j=1}^{n} L_j = \frac{L_n + L_{n+1} - k - 2}{k}$, which can be proved by induction or by using the Binet's formula for k-Lucas numbers.

Inequality (ii) follows by Jensen's inequality as before, and using that $\displaystyle\sum_{j=1}^{n} L_j^2 = \frac{L_{2n+1}}{k} + (-1)^n - 2$, which may be proved by induction or by using the Binet's formula for k-Lucas