Therefore,

$$\begin{aligned}\sum_{k=1}^{n} G_k^3 G_{k+1}^3 &= \sum_{k=1}^{n}\left[\left(\frac{1}{2}\,G_k G_{k+1} G_{k+2}\right)^2 - \left(\frac{1}{2}\,G_{k-1} G_k G_{k+1}\right)^2\right]\\ &= \left(\frac{1}{2}\,G_n G_{n+1} G_{n+2}\right)^2 - \left(\frac{1}{2}\,G_0 G_1 G_2\right)^2\\ &= \left(\sum_{k=1}^{n} G_k^2 G_{k+1} + \frac{1}{2}\,G_0 G_1 G_2\right)^2 - \left(\frac{1}{2}\,G_0 G_1 G_2\right)^2\\ &= \left(\sum_{k=1}^{n} G_k^2 G_{k+1}\right)^2 + G_0 G_1 G_2 \sum_{k=1}^{n} G_k^2 G_{k+1}.\end{aligned}$$

Editor's Note: This problem is a Lucas analog of Problem B-1136, Volum 51.4 (2013).

Also solved by Michel Bataille, Brian Bradie, I. V. Fedak, Dmitry Fleischman, Robert Frontczak, Wei-Kai Lai and John Risher (student) (jointly), Ehren Metcalfe (computer proof), Ángel Plaza, Raphael Schumacher (student), Albert Stadler, David Terr, Daniel Văcaru, and the proposer.

An Inequality Derived from the Trapezoidal Rule

B-1248 Proposed by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain.
(Vol. 57.2, May 2019)

For all positive integers n and a, prove that

$$\sum_{k=0}^{n} L_k(L_{k+1}^a + L_{k+2}^a) \le (L_{n+2} - 1)(L_{n+2}^a + 1).$$

Solution by Brian Bradie, Christopher Newport University, Newport News, VA.

Replacing L_k with $L_{k+2} - L_{k+1}$, noting $L_1 = 1$, and dividing by 2, we see that the desired inequality is equivalent to

$$\sum_{k=0}^{n}(L_{k+2} - L_{k+1})\,\frac{L_{k+1}^a + L_{k+2}^a}{2} \le (L_{n+2} - L_1)\,\frac{L_{n+2}^a + L_1^a}{2}.$$

The summation on the left side of this inequality is a trapezoidal rule approximation of the value of

$$\int_{L_1}^{L_{n+2}} x^a\,dx$$

using the $n+1$ subintervals $[L_1, L_2], [L_2, L_3], [L_3, L_4], \ldots, [L_{n+1}, L_{n+2}]$, whereas the expression on the right side is a trapezoidal rule approximation to the value of the same integral using just one subinterval $[L_1, L_{n+2}]$. Because $f(x) = x^a$ is convex on $[L_1, L_{n+2}]$ for all positive integers n and a, the desired inequality follows immediately.

Moreover, because $f(x) = x^a$ is convex on $[L_1, L_{n+2}]$ for all positive integers n and all real numbers $a \ge 1$ or $a \le 0$, the inequality holds for all real numbers $a \ge 1$ or $a \le 0$, with equality