Solution by Jason L. Smith, Richland Community College, Decatur, IL.

Let γ represent either α or β. Consider the sum

$$\sum_{k=1}^{n}\binom{n}{k}\gamma^k(x+1)^k = [1+\gamma(x+1)]^n = (\gamma+1+\gamma x)^n.$$

Since $\gamma^2 = \gamma + 1$, we obtain

$$(\gamma+1+\gamma x)^n = (\gamma^2+\gamma x)^n = \gamma^n(\gamma+x)^n = \gamma^n\sum_{k=0}^{n}\binom{n}{k}\gamma^{n-k}x^k.$$

Therefore,

$$\sum_{k=1}^{n}\binom{n}{k}\gamma^{m+k}(x+1)^k = \sum_{k=0}^{n}\binom{n}{k}\gamma^{m+2n-k}x^k.$$

Because this identity is satisfied using $\gamma = \alpha$ or $\gamma = \beta$, it is also satisfied by any generalized Fibonacci sequence expressible as $G_r = a\alpha^r + b\beta^r$. Thus, we can claim in general that

$$\sum_{k=0}^{n}\binom{n}{k}G_{m+k}(x+1)^k = \sum_{k=0}^{n}\binom{n}{k}G_{m+2n-k}x^k,$$

which proves both identities.

Editor's Notes: Based on Frontczak's observation, it is easy to further extend the result to $\sum_{k=0}\binom{n}{k}G_{m+k}(x+p)^k = \sum_{k=0}^{n}\binom{n}{k}G_{m+2n-k}(x+p-1)^k$ for any real number p.

Also solved by Ulrich Abel, Michel Bataille, Khristo N. Boyadzhiev, Brian Bradie, I. V. Fedak, Dmitry Fleischman, Robert Frontczak, Hideyuki Ohtsuka, Raphael Schumacher (student), Albert Stadler, David Terr, and the proposer.

Catalan and an Old Elementary Problem

B-1252 Proposed by Hideyuki Ohtsuka, Saitama, Japan. (Vol. 57.3, August 2019)

For integers $n \geq 0$ and $r \geq 0$, prove that

$$F_{n-r}F_nF_{n+r} + \sum_{k=1}^{n+1}F_{k-r}F_kF_{k+r} = \frac{F_{3n+2}+1}{2} + (-1)^rF_r^2.$$

Solution by Steve Edwards, Roswell, GA.

We use two identities from [1]: the Catalan identity (page 106)

$$F_{k-r}F_{k+r} = F_k^2 + (-1)^{k+r+1}F_r^2,$$

and

$$\sum_{k=1}^{n+1}(-1)^{k+1}F_k = 1 + (-1)^nF_n.$$

We also use the identity from Problem B-1211 from this *Quarterly* [4]

$$\sum_{k=1}^{n+1}F_k^3 = \frac{F_{3n+2}+1}{2} - F_n^3.$$

We have

$$\begin{aligned}\sum_{k=1}^{n+1} F_{k-r}F_kF_{k+r} &= \sum_{k=1}^{n+1} F_k\left[F_k^2+(-1)^{k+r+1}F_r^2\right]\\ &= \sum_{k=1}^{n+1} F_k^3+(-1)^rF_r^2\sum_{k=1}^{n+1}(-1)^{k+1}F_k\\ &= \frac{F_{3n+2}+1}{2}-F_n^3+(-1)^rF_r^2\left[1+(-1)^nF_n\right]\\ &= \frac{F_{3n+2}+1}{2}+(-1)^rF_r^2-F_n\left[F_n^2+(-1)^{n+r+1}F_r^2\right]\\ &= \frac{F_{3n+2}+1}{2}+(-1)^rF_r^2-F_{n-r}F_nF_{n+r}.\end{aligned}$$

This completes the proof.

Editor's Notes: Using the following identities [2, 3]

$$\begin{aligned}L_{n+r}L_{n-r}-L_n^2 &= 5(-1)^{n+r}F_r^2,\\ 2\sum_{k=0}^{n} L_k^3+2L_{n-1}^3 &= 5L_{3n-1}+19,\\ \sum_{k=1}^{n+1}(-1)^kL_k &= (-1)^{n+1}L_n+1.\end{aligned}$$

Koshy derived the Lucas analog

$$L_{n-r}L_nL_{n+r}+\sum_{k=1}^{n+1} L_{k-r}L_kL_{k+r}=\frac{5L_{3n+2}+3}{2}+5(-1)^rF_r^2.$$

References

[1] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley & Sons, New York, 2001.

[2] T. Koshy, *Fibonacci and Lucas Numbers with Applications, Vol. II*, John Wiley & Sons, New York, 2019.

[3] T. Koshy and Z. Gao, *Extended Gibonacci sums of polynomial products of order 3 revisited*, The Fibonacci Quarterly, to appear.

[4] H. Ohtsuka, *Problem B-1211*, The Fibonacci Quarterly, **55.3** (2017), 276.

Also solved by Michel Bataille, Brian Bradie, Steve Edwards (second solution), I. V. Fedak, Dmitry Fleischman, Robert Fontczak, Thomas Koshy, Ángel Plaza, Raphael Schumacher (student), Jason L. Smith, and the proposer.

A Trigonometric Inequality

B-1253 D. M. Bătineţu-Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade School, Buzău, Romania.
(Vol. 57.3, August 2019)

Prove that

$$\sin F_{2n+2}+\sin F_n^2+\cos F_{n+2}^2\le\frac{3}{2}.$$