Another Solution Using Jensen's Inequality

B-1255 Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain. (Vol. 57.3, August 2019)

For any positive integer n, prove that it holds:

$$\sqrt{\left(\frac{F_n-1}{F_n}\right)^{F_n}\left(\frac{L_n-1}{L_n}\right)^{L_n}} \leq \left(\frac{F_{n+1}-1}{F_{n+1}}\right)^{F_{n+1}}.$$

Solution by Albert Stadler, Herrliberg, Switzerland.

The function $f(x) = x\log\left(1-\frac{1}{x}\right)$ is concave for $x > 1$, since $f''(x) = -\frac{1}{x(x-1)^2} < 0$. Therefore, by Jensen's inequality,

$$f(x)+f(y) \leq 2\,f\left(\frac{x+y}{2}\right).$$

Hence,

$$x\,\log\left(1-\frac{1}{x}\right)+y\,\log\left(1-\frac{1}{y}\right) \leq (x+y)\log\left(1-\frac{2}{x+y}\right)$$

for all $x, y > 1$. We put $x = F_n$, $y = L_n$. Then, $F_n + L_n = 2F_{n+1}$, and the statement follows for $n > 2$. The statement holds trivially true for $n = 1$ and $n = 2$.

Also solved by Michel Bataille, Brian Bradie, I. V. Fedak, Dmitry Fleischman, Robert Frontczak, Hideyuki Ohtsuka, Ángel Plaza, David Terr, and the proposer.