Reducing the Balancing Numbers Modulo n

B-1254 Proposed by Robert Frontczak, Landesbank Baden-Württemberg, Stuttgart, Germany.
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Let B_n be the nth balancing number defined as $B_0 = 0$, $B_1 = 1$, and $B_n = 6B_{n-1} - B_{n-2}$ for $n \geq 2$. Show that

$$B_n + 4\sum_{k=0}^{n} kB_k \equiv 0 \pmod{n}$$

for all positive integers n.

Solution by Brian Bradie, Christopher Newport University, Newport News, VA.

We will show that

$$B_n + 4\sum_{k=0}^{n} kB_k = n(B_{n+1} - B_n)$$

for all positive integers n, from which the desired congruence follows immediately. The identity is easy to verify when $n = 1$. Assume

$$B_n + 4\sum_{k=0}^{n} kB_k = n(B_{n+1} - B_n)$$

for some positive integer n. Then,

$$\begin{aligned} B_{n+1} + 4\sum_{k=0}^{n+1} kB_k &= (4n+5)B_{n+1} + 4\sum_{k=0}^{n} kB_k \\ &= (4n+5)B_{n+1} - B_n + n(B_{n+1} - B_n) \\ &= (n+1)(5B_{n+1} - B_n) \\ &= (n+1)(6B_{n+1} - B_n - B_{n+1}) \\ &= (n+1)(B_{n+2} - B_{n+1}). \end{aligned}$$

The identity follows by induction.

Editor's Notes: Davenport noticed that for the generalized balanced numbers with $B_0 = a$ and $B_1 = b$, the congruence becomes

$$B_n + 4\sum_{k=1}^{n} kB_k \equiv a \pmod{n}.$$

Going in a different direction, Fedak observed that

$$C_n + (p-2)\sum_{k=0}^{n} kC_k = n(C_{n+1} - C_n) \equiv 0 \pmod{n},$$

where $C_0 = 0$, $C_1 = 1$, and $C_n = p\,C_{n-1} - C_{n-2}$ for $n \geq 2$.

Also solved by Michel Bataille, Kenny B. Davenport, I. V. Fedak, Dmitry Fleischman, Ernest James (student), Ángel Plaza, Raphael Schumacher (student), Jason L. Smith, David Terr, and the proposer.