H-862 Proposed by Ángel Plaza, Gran Canaria, Spain

Let $(F_{k,n})_{n\in\mathbb{Z}}$ and $(L_{k,n})_{n\in\mathbb{Z}}$ denote the k-Fibonacci and k-Lucas numbers given by $F_{k,n+1} = kF_{k,n} + F_{k,n-1}$, $L_{k,n+1} = kL_{k,n} + L_{k,n-1}$ for $n \geq 1$ with $F_{k,0} = 0$, $F_{k,1} = 1$, $L_{k,0} = 2$, $L_{k,1} = k$. Prove that for integers $m \geq 1$ and $j \geq 0$ we have

$$\text{(i)} \quad \sum_{n=1}^{m} F_{k,n\pm j}L_{k,n\mp j} = \frac{F_{k,2m+1}-1}{k} + \begin{cases} 0, & \text{if } m \equiv 0 \pmod 2; \\ (-1)^j F_{k,\pm 2j}, & \text{if } m \equiv 1 \pmod 2. \end{cases}$$

$$\text{(ii)} \quad \sum_{n=1}^{m} F_{k,n+j}F_{k,n-j}L_{k,n+j}L_{k,n-j} = \frac{F_{k,4m+2}/k - 1 - mL_{k,4j}}{k^2+4}.$$

SOLUTIONS

A circular inequality

H-825 Proposed by D. M. Bătineţu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania (Vol. 56, No. 3, August 2018)

If $a, b, c > 0$ and n is a positive integer, prove that

$$2\left(\left(\frac{a}{bF_n + F_{n+1}c}\right)^3 + \left(\frac{b}{F_nc + F_{n+1}a}\right)^3 + \left(\frac{c}{F_na + F_{n+1}b}\right)^3\right)$$
$$+ \quad 3\frac{abc}{(F_na + F_{n+1}b)(F_nb + F_{n+1}c)(F_nc + F_{n+1}a)} \geq \frac{9}{F_{n+2}^3}.$$

Solution by Wei-Kai Lai, University of South Carolina Salkehatchie, Walterboro, SC

Let

$$x = \frac{a}{bF_n + F_{n+1}c}, \quad y = \frac{b}{cF_n + aF_{n+1}}, \quad z = \frac{c}{aF_n + bF_{n+1}}.$$

According to Surányi's inequality ([1], Theorem 4.4):

$$2\left(x^3 + y^3 + z^3\right) + 3xyz \geq (x+y+z)(x^2+y^2+z^2).$$

Because the quadratic mean is greater than or equal to the arithmetic mean, it is easy to check that

$$x^2 + y^2 + z^2 \geq \frac{1}{3}(x+y+z)^2.$$

So, we only need to prove that

$$\frac{1}{3}(x+y+z)^3 \geq \frac{9}{F_{n+2}^3},$$

or equivalently

$$x + y + z \geq \frac{3}{F_{n+2}}.$$

According to Radon's inequality,

$$
\begin{aligned}
x+y+z &= \frac{a^2}{F_nab+F_{n+1}ac}+\frac{b^2}{F_nbc+F_{n+1}ab}+\frac{c^2}{F_nac+F_{n+1}bc}\\
&\geq \frac{(a+b+c)^2}{(F_nab+F_{n+1}ac)+(F_nbc+F_{n+1}ab)+(F_nac+F_{n+1}bc)}\\
&= \frac{(a+b+c)^2}{(F_n+F_{n+1})(ab+ac+bc)} \geq \frac{3(ab+ac+bc)}{F_{n+2}(ab+ac+bc)}=\frac{3}{F_{n+2}},
\end{aligned}
$$

hence proving the claimed inequality. The equality holds when $a=b=c$.

[1] Z. Cvetkovski, *Inequalities, Theorems, Techniques and Selected Problems*, Springer, New York, 2012, p. 35.

Also solved by Brian Bradie, Dmitry Fleischman, Ángel Plaza, Nicuşor Zlota, and the proposers.

Powers of 2 and powers of 3

H-826 Proposed by Hideyuki Ohtsuka, Saitama, Japan (Vol. 56, No. 3, August 2018)

For an integer $n \geq 0$, prove that

$$
\sum_{\substack{a+b=n\\ a,b\geq 0}} \frac{1}{L_{2^a3^b}F_{2^a3^{b+1}}} = \frac{F_{3^{n+1}-2^{n+1}}}{F_{3^{n+1}}F_{2^{n+1}}}.
$$

Solution by the proposer

We use the identities

(1) $F_{s-t}=(-1)^t(F_{t+1}F_s-F_tF_{s+1})$ (see [1] (9));
(2) $F_{2s}=F_sL_s$ (see [1] (13)).

We have

$$
\begin{aligned}
\frac{F_{2^{a+1}3^b+1}}{F_{2^{a+1}3^b}}-\frac{F_{2^a3^{b+1}+1}}{F_{2^a3^{b+1}}} &= \frac{F_{2^{a+1}3^b+1}F_{2^a3^{b+1}}-F_{2^{a+1}3^b}F_{2^a3^{b+1}+1}}{F_{2^{a+1}3^b}F_{2^a3^{b+1}}}\\
&= \frac{F_{2^a3^{b+1}-2^{a+1}3^b}}{F_{2^{a+1}3^b}F_{2^a3^{b+1}}} \text{(by (1))} = \frac{F_{2^a3^b}}{F_{2^{a+1}3^b}F_{2^a3^{b+1}}} = \frac{1}{L_{2^a3^b}F_{2^a3^{b+1}}} \text{ (by (2))}.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
\sum_{\substack{a+b=n\\ a,b\geq 0}} \frac{1}{L_{2^a3^b}F_{2^a3^{b+1}}} &= \sum_{\substack{a+b=n\\ a,b\geq 0}} \left(\frac{F_{2^{a+1}3^b+1}}{F_{2^{a+1}3^b}}-\frac{F_{2^a3^{b+1}+1}}{F_{2^a3^{b+1}}}\right)\\
&= \sum_{a=0}^{n} \left(\frac{F_{2^{a+1}3^{n-a}+1}}{F_{2^{a+1}3^{n-a}}}-\frac{F_{2^a3^{n-a+1}+1}}{F_{2^a3^{n-a+1}}}\right) = \frac{F_{2^{n+1}+1}}{F_{2^{n+1}}}-\frac{F_{3^{n+1}+1}}{F_{3^{n+1}}}\\
&= \frac{F_{2^{n+1}+1}F_{3^{n+1}}-F_{2^{n+1}}F_{3^{n+1}+1}}{F_{2^{n+1}}F_{3^{n+1}}} = \frac{F_{3^{n+1}-2^{n+1}}}{F_{2^{n+1}}F_{3^{n+1}}} \text{ (by (1))}.
\end{aligned}
$$

[1] S. Vajda, *Fibonacci and Lucas numbers and the Golden Section*, Dover, 2008.

Also solved by Brian Bradie and Dmitry Fleischman.