for $a = 1$ and $a = 0$. For $0 < a < 1$, the function $f(x) = x^a$ is concave on $[L_1, L_{n+2}]$, so the opposite inequality holds; that is,

$$\sum_{k=0}^{n}(L_{k+2} - L_{k+1})\,\frac{L_{k+1}^a + L_{k+2}^a}{2} > (L_{n+2} - L_1)\,\frac{L_{n+2}^a + L_1^a}{2}.$$

Editor's Note: This problem is a Lucas analog of Problem B-1223, Volume 56.1 (2018).

Also solved by Dmitry Fleischman, Wei-Kai Lai, I. V. Fedak, and the proposer.

The Tails of Two Series

B-1249 Proposed by Hideyuki Ohtsuka, Saitama, Japan. (Vol. 57.2, May 2019)

For positive integers s and t, prove that

$$\sum_{n=s}^{\infty} \frac{(-1)^{tn}}{\alpha^{(2t-1)n} F_n} = \sum_{n=t}^{\infty} \frac{(-1)^{sn}}{\alpha^{(2s-1)n} F_n}.$$

Solution by I. V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.

Let $q = \frac{\beta}{\alpha}$, so that $|q| < 1$. Using $\alpha\beta = -1$, we obtain

$$\sum_{n=s}^{\infty} \frac{(-1)^{tn}}{\alpha^{(2t-1)n} F_n} = \sqrt{5}\sum_{n=s}^{\infty}\left(\frac{\beta}{\alpha}\right)^{tn}\frac{\alpha^n}{\alpha^n - \beta^n} = \sqrt{5}\sum_{n=s}^{\infty}\frac{q^{tn}}{1-q^n} = \sqrt{5}\sum_{n=s}^{\infty} q^{tn}\sum_{k=0}^{\infty} q^{kn}.$$

Similarly,

$$\sum_{n=t}^{\infty} \frac{(-1)^{sn}}{\alpha^{(2s-1)n} F_n} = \sqrt{5}\sum_{n=t}^{\infty} q^{sn}\sum_{k=0}^{\infty} q^{kn}.$$

Because

$$\begin{aligned}\sum_{n=s}^{\infty} q^{tn}\sum_{k=0}^{\infty} q^{kn} &= \sum_{n=s}^{\infty}\sum_{k=0}^{\infty} q^{(t+k)n} = \sum_{\ell=0}^{\infty}\sum_{k=0}^{\infty} q^{(t+k)(s+\ell)} \\ &= \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} q^{(s+\ell)(t+k)} = \sum_{n=t}^{\infty}\sum_{\ell=0}^{\infty} q^{(s+\ell)n} = \sum_{n=t}^{\infty} q^{sn}\sum_{\ell=0}^{\infty} q^{\ell n},\end{aligned}$$

we have that

$$\sum_{n=s}^{\infty} \frac{(-1)^{tn}}{\alpha^{(2t-1)n} F_n} = \sum_{n=t}^{\infty} \frac{(-1)^{sn}}{\alpha^{(2s-1)n} F_n}.$$

Also solved by Michel Bataille, Raphael Schumacher (student), and the proposer.

Make It Telescopic!

B-1250 Proposed by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain. (Vol. 57.2, May 2019)