B-1257 Proposed by Robert Frontczak, Landesbank Baden-Württemberg, Stuttgart, Germany.

Find closed form expressions for the alternating sums

$$\sum_{k=0}^{n}(-1)^k F_{3^k}F_{2\cdot 3^k} \qquad \text{and} \qquad \sum_{k=0}^{n}(-1)^k F_{3^k}L_{2\cdot 3^k}.$$

B-1258 Proposed by D. M. Bătineţu-Giurgiu, Mateo Basarab National College, Bucharest, Romania, and Neculai Stanciu, George Emil Palade School, Buzău, Romania.

Prove that

(i) $\sin\left(F_{2n+3}\right)+\sin\left(F_{n+1}F_n\right)+\cos\left(F_{n+3}F_{n+2}\right)\le\frac{3}{2}$

(ii) $\sin\left(F_mL_n\right)+\sin\left(F_nL_m\right)+\cos\left(2F_{m+n}\right)\le\frac{3}{2}$

B-1259 Proposed by Ángel Plaza and Sergio Falcón, Universidad de Las Palmas de Gran Canaria, Spain.

Let k be a positive integer. The k-Fibonacci numbers are defined by the recurrence relation $F_{k,n+1}=kF_{k,n}+F_{k,n-1}$, with initial values $F_{k,0}=0$ and $F_{k,1}=1$. Prove that

(i) $\displaystyle\sum_{i=1}^{n}\frac{F_{k,i}^2}{\sqrt{F_{k,i}+1}}\ge\frac{(F_{k,n}+F_{k,n+1}-1)^2}{k\sqrt{kn(F_{k,n}+F_{k,n+1}-1+kn)}}$

(ii) $\displaystyle\sum_{i=1}^{n}\frac{F_{k,i}^4}{\sqrt{F_{k,i}^2+1}}\ge\frac{F_{k,n}^2F_{k,n+1}^2}{k\sqrt{kn(F_{k,n}F_{k,n+1}+kn)}}$

B-1260 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

For any positive integer n, find a closed form expression for the sum

$$\sum_{k=1}^{n}\left\lfloor\frac{F_k}{\alpha F_k-F_{k+1}}\right\rfloor.$$

SOLUTIONS

An Easy Consequence of Radon's Inequality

B-1236 Proposed by D. M. Bătineţu-Giurgiu, Matei Basarab National College, Bucharest, Romania, and Neculai Stanciu, George Emil Palade School, Buzău, Romania.

Prove that, for any integers $m\ge 0$ and $n>1$,

$$\sum_{k=1}^{n+1}\frac{\binom{n}{k-1}^{m+1}}{F_k^{2m}}>\frac{2^{n(m+1)}}{F_{n+1}^mF_{n+2}^m}, \qquad \text{and} \qquad \sum_{k=1}^{n+1}\frac{F_k^{m+1}}{\binom{n}{k-1}^m}>\frac{(F_{n+3}-1)^{m+1}}{2^{mn}}.$$

Editor's Note: The inequalities become equalities when $m=0$; the condition on m should have been $m>0$.

Solution by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

The inequalities follow by applying Radon's inequality, which asserts that if $x_k, a_k > 0$ for $k \in \{1, 2, \ldots, n\}$ and $m > 0$, then

$$\frac{x_1^{m+1}}{a_1^m} + \frac{x_2^{m+1}}{a_2^m} + \cdots + \frac{x_n^{m+1}}{a_n^m} \geq \frac{(x_1 + x_2 + \cdots + x_n)^{m+1}}{(a_1 + a_2 + \cdots + a_n)^m},$$

in which equality is attained when $\frac{x_1}{a_1} = \frac{x_2}{a_2} = \cdots = \frac{x_n}{a_n}$. In our case, it leads us to

$$\sum_{k=1}^{n+1} \frac{\binom{n}{k-1}^{m+1}}{F_k^{2m}} > \frac{\left[\sum_{k=1}^{n+1} \binom{n}{k-1}\right]^{m+1}}{\left(\sum_{k=1}^{n+1} F_k^2\right)^m} = \frac{2^{n(m+1)}}{F_{n+1}^m F_{n+2}^m},$$

because $\sum_{k=1}^{n+1} F_k^2 = F_{n+1}F_{n+2}$.

For the second inequality, we have

$$\sum_{k=1}^{n+1} \frac{F_k^{m+1}}{\binom{n}{k-1}^m} > \frac{\left(\sum_{k=1}^{n+1} F_k\right)^{m+1}}{\left[\sum_{k=1}^{n+1} \binom{n}{k-1}\right]^m} = \frac{(F_{n+3} - 1)^{m+1}}{2^{nm}},$$

because $\sum_{k=1}^{n+1} F_k = F_{n+3} - 1$.

Also solved by Michel Bataille, Brian Bradie, I. V. Fedak, Dmitry Fleischman, Hideyuki Ohtsuka, Wei-Kai Lai and John Risher (student) (jointly), Henry Ricardo, Anthony Vasaturo, and the proposers.

A Telescoping Product

B-1237 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

Evaluate

$$\prod_{k=1}^{\infty} \left(1 + \frac{1}{\alpha^k + \alpha}\right), \qquad \text{and} \qquad \prod_{k=1}^{\infty} \left(1 - \frac{1}{\alpha^k + \alpha}\right).$$

Solution by Steve Edwards, Kennesaw State University, Marietta, GA.

We will use the following: $\alpha^2 = \alpha + 1$, $\alpha - 1 = \alpha^{-1}$, and $\lim_{n\to\infty} \frac{1}{\alpha^n} = 0$. The kth factor in the first product is

$$1 + \frac{1}{\alpha^k + \alpha} = \frac{\alpha^k + \alpha + 1}{\alpha^k + \alpha} = \frac{\alpha^k + \alpha^2}{\alpha^k + \alpha} = \frac{\alpha(\alpha^{k-2} + 1)}{\alpha^{k-1} + 1}.$$

This gives

$$\prod_{k=1}^{n} \left(1 + \frac{1}{\alpha^k + \alpha}\right) = \alpha^n \prod_{k=1}^{n} \frac{\alpha^{k-2} + 1}{\alpha^{k-1} + 1} = \frac{\alpha^n(\alpha^{-1} + 1)}{\alpha^{n-1} + 1} = \frac{\alpha^{n+1}}{\alpha^{n-1} + 1} = \frac{\alpha^2}{1 + \alpha^{-n+1}}.$$

Then

$$\prod_{k=1}^{\infty} \left(1 + \frac{1}{\alpha^k + \alpha}\right) = \lim_{n\to\infty} \frac{\alpha^2}{1 + \alpha^{-n+1}} = \alpha^2.$$

Next,

$$1 - \frac{1}{\alpha^k + \alpha} = \frac{\alpha^k + \alpha - 1}{\alpha^k + \alpha} = \frac{\alpha^k + \alpha^{-1}}{\alpha^k + \alpha} = \frac{1}{\alpha^2} \cdot \frac{\alpha^{k+1} + 1}{\alpha^{k-1} + 1}.$$