approximately. When $\theta \approx 294^\circ$, it is approximately $0.41 \frac{\pi R^3}{3}$.

In conclusion, if we want to maximise the total volume of the two cones then we cut out a sector with central angle approximately $115.2^\circ$ from the given metal sheet and roll the two sectors into right circular cones.

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106.07 A function-based proof of the harmonic mean – geometric mean – arithmetic mean inequalities

For $a, b \in \mathbb{R}$, with $0 < b \leq a$, the harmonic, geometric and arithmetic means of $a$ and $b$ are respectively defined by

$$H(a, b) = \frac{2}{\frac{1}{a} + \frac{1}{b}} = \frac{2ab}{a+b}, \quad G(a, b) = \sqrt{ab} \quad \text{and} \quad A(a, b) = \frac{a+b}{2}.$$

**Theorem:** For $0 < b \leq a$, $H(a, b) \leq G(a, b) \leq A(a, b)$, that is

$$\frac{2ab}{a+b} \leq \sqrt{ab} \leq \frac{a+b}{2}.$$

**Proof:** If $x = \frac{b}{a}$, then $x \in (0, 1]$ and the inequalities to prove are

$$\frac{2x}{1 + x} \leq \sqrt{x} \leq \frac{1 + x}{2}.$$

There are easy purely algebraic proofs for these inequalities [1]. Here, instead, we propose an elementary approach based on the graph of some functions to prove them.

![Graph](https://www.cambridge.org/core/terms.https://doi.org/10.1017/mag.2022.22)
From now on, we use the notation $H = H(a, b)$, $G = G(a, b)$ and $A = A(a, b)$.

**Step 1**: $G \leq A$ and $G = A$ if, and only if, $a = b$, that is $\sqrt{x} \leq \frac{1 + x}{2}$ for $x \in (0, 1]$, with equality only for $x = 1$.

If $f(x) = \frac{1}{2}(1 + x)$, and $g(x) = \sqrt{x}$, then $f(0) = \frac{1}{2}$, $g(0) = 0$, $f(1) = g(1) = 1$. In addition, $f'(x) = \frac{1}{2}$, while $g'(x) = 1/(2\sqrt{x})$. Since $g'(x) > \frac{1}{2}$, for $x \in (0, 1)$, with $g'(1) = \frac{1}{2}$, it follows that $g(x) < f(x)$ for $x \in (0, 1)$, and the proof is complete.

**Step 2**: Note that $G(A, H) = G$, because $\sqrt{x} = \sqrt{\frac{2x}{1 + x} \cdot \frac{1 + x}{2}}$. Since $G \leq A$, then $H \leq G \leq A$, as the following figure shows:

**Reference**