REV. DE GEOFISICA (1992) **48**, 159-164

A simple modelling of braid-like structures (*rill marks*) appearing on sandy beaches

J. M. PACHECO

Departamento de Matemáticas. Facultad de Ciencias del Mar. Universidad de Las Palmas de Gran Canaria

RESUMEN

Se presenta un modelo sencillo para la generación de estructuras dendríticas trenzadas (*rill marks*) que se observan en playas arenosas tras la bajada de la marea, y que son debidas al flujo de agua que rezuma de la arena saturada. El modelo consiste en un sistema no lineal con término de ruido aditivo. Estéticamente, los *rill marks* se consideran como la superposición de varias realizaciones del proceso estocástico solución del sistema.

ABSTRACT

A simple approach to modelling *rill marks* patterns (braid-like structures appearing in the surface of beaches due to water oozing after high water) is presented. The model discussed is a numerically treated stochastic nonlinear system, upon which adequate conditions are imposed. Aesthetically, *rill marks* structures are considered as superposition of several realizations of the stochastic process.

INTRODUCTION

Often one can observe on a flat, sandy, beach that some time after high tide, a thin water film gives rise to erratic flows tracing wavy figures on the sand. This water film is the result of upwelling due mainly to settling of the sand bed when the tidal level decreases. The resulting patterns are the effects of various sedimentary transport processes of sand particles showing complicated, braidlike intertwining figures, known as *rill marks*. These structures have been described long ago (Williamson, 1887, cited in Allen [1986]) and several mechanisms for their formation are available. Nevertheless no mathematical model seems to have been developped. See Figure 1.

Rill marks are interesting examples of complex nonlinear behaviour in natural systems, and they deserve being studied through techniques employed for the analysis of oscillations. The typical spatial and temporal scales for rill marks are meters and seconds, and after formation they remain long enough to be easily observable. River meandering is another example of this complexity, and its nonlinear nature can be conjectured from field observations: The similarity between the aerial photograph in Allen (1986), page 55, vol. 2, and the solution of a Van der Pol oscillator is astonishing. See Figure 2.

Here the following physical ideas are considered for modelling purposes:

1. Water flow oozes out of the saturated sand and flows upon an erodible bed under the action of gravity. Here erodible means that the trajectory of the flux and transported sand particles remains marked on the bed, without taking into account the actual sediment transport mechanism. This assumption is geometrically simple and convenient, though

Fig. 1.—*Rill Marks.* Playa de Maspalomas, Gran Canaria, December 1991.

particle transport is determinant in carving the runoff bed.

Normally one can observe a tree-like dendritic form whose branches are small streams of water converging downstream to a common trunk (Figure 3), where usually an overall braid-like pattern is very visible. Only rarely (Pettijohn [1964]) downstream amplification can also be observed, so it is not considered here. Sometimes different densities and colours of the sediment mark the general macroscopic structure.

2. Once a water flow starts its way, it undergoes complex mechanical interactions with sand particles: Thus the trajectory is deviated and oscillations around an ideal rectilinear path appear, as well as possible drifts depending on the curvature of the beach surface and on the saturation level of the wet sand. This approach by means of tracking singular trajectories has been used by the author elsewhere (Pacheco and Fernández [1988]). These trajectories are the elements of the overall observable braid-like patterns.

The above considerations can be translated into a simple mathematical model in order to generate patterns that reproduce the observed figures. The orthogonal distance of the actual flow filament to the ideal rectilinear path (i. e. the axis of the final trunk) is taken as the relevant variable. Nevertheless, in some modelling techniques the fundamental variable is the angle the stream deviates from the ideal axis (Anderson 1988).

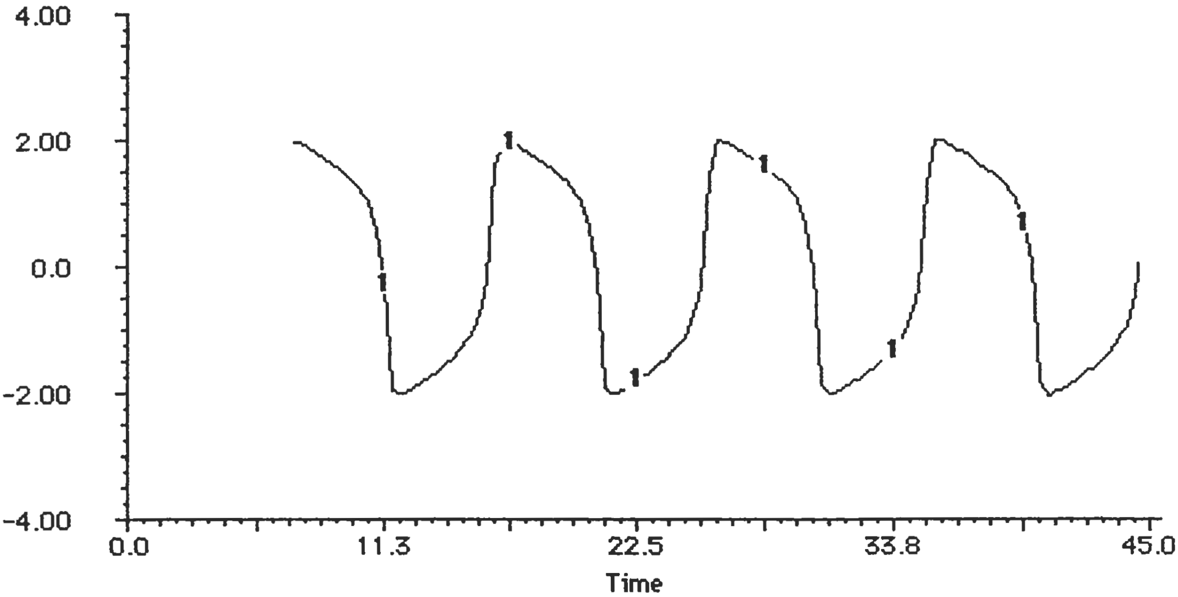
MODELLING

Adding a multiple of the first derivative x' to the simple harmonic oscillator

$$x'' + \omega^2 x = 0$$

(here the natural frequency ω models macroscopic features) amounts to considering microscopic features of the bed, which account for friction effects. Among these features are grain size and shape and fractal characteristics of individual sand particles, as well as saturation of the sand bed. The individual effects of these causes are difficult to separate and quantify.

For the coefficient of x' there is a large choice; the simplest one is the linear case where it reduces to a



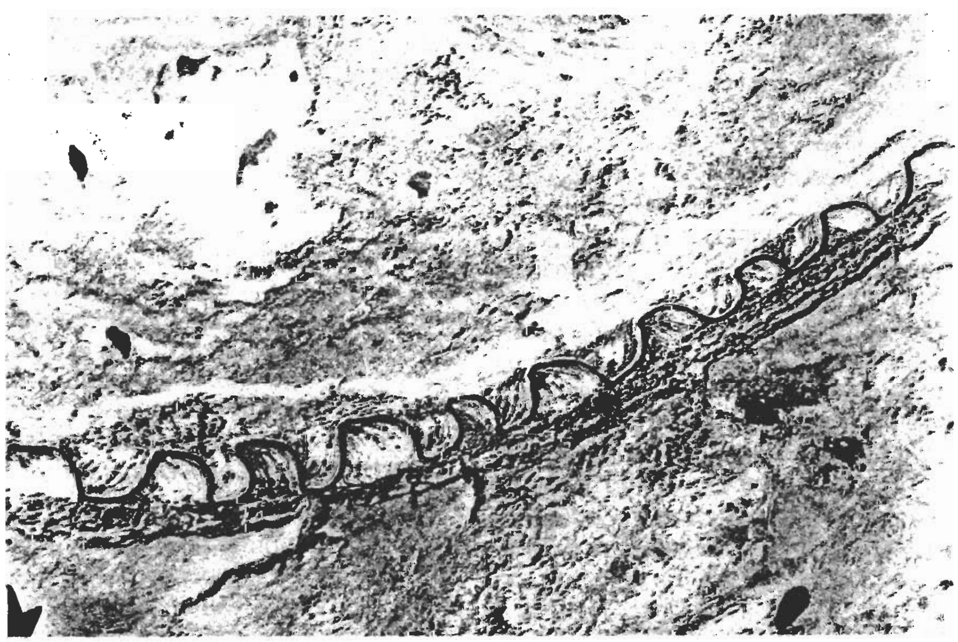


Fig. 2.—River meandering and the Van der Pol oscillator.

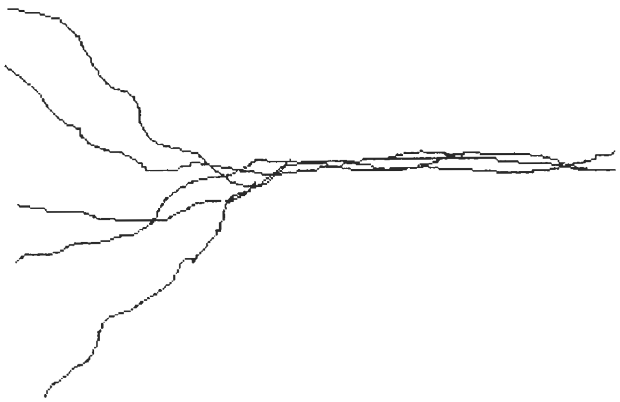
numerical value α. Appropriate combinations of α and ω give oscillatory patterns with amplification or with decay to 0. For this modelling a decaying pattern will be chosen.

A nonlinear choice for the coefficient of x' of the type $f(x, x')$ is known to produce oscillatory behaviours of a more complex nature (see e. g. Jordan and Smith [1987]), as in the case of the above cited Van der Pol oscillator.

In any case we consider a Liénard type equation:

$$x'' + f(x, x')x' + \omega^2 x = G(t)$$

where the forcing term $G(t)$ represents environmental



← 70- 100 cm →

Fig. 3.—Schematic representation of a typical *rill mark*.

conditions acting on the system, generally at scales too small to be described with standard measurements, so they will be considered as *noise*. The above equation in thus equivalent to the system

$$x' = y$$

$$y' = -\omega^2 x - f(x, y)y + \varepsilon(t)$$

where $\varepsilon(t) = \zeta\xi(t)$. Here ζ is a measure of noise intensity and $\xi(t)$ is Gaussian white noise. With this modelling scheme the trajectories are considered as realizations of the particular stochastic process solution of the nonlinear stochastic system. The idea behind it is that the whole observable system can be built up by considering a number of these realizations and superimposing them.

The damping function $f(x, y)$ will not, in general, satisfy conditions for the system to be Hamiltonian. This prevents a simple theoretical analysis, so numerical modelling will be employed in the study of these patterns.

For the noiseless or deterministic system:

$$x' = y$$

$$y' = -\omega^2 x - f(x, y)y$$

the only equilibrium point is the origin, where the eigenvalues of the linearized system are the roots of the quadratic equation:

$$\lambda^2 + \lambda f(0, 0) + \omega^2 = 0$$

Damped oscillations appear if $f(0, 0) > 0$ and if the discriminant of the quadratic is also negative. This happens whenever the natural frequency satisfies the inequality $\omega > \frac{f(0,0)}{2}$. If this last inequality is reversed, the system trajectories decay to the origin without oscillation. The attraction basin of the origin will be bounded by a limit cycle, whose existence is guaranteed if $f(x, y)$ and the linear term $\omega^2 x$ satisfy the hypothesis of the Bendixson criterion. Otherwise the basin of attraction is the whole phase plane.

With the introduction of the additive noise $\varepsilon(t)$ (Langevin type problem) the trajectories are now determined via the probability density function obtained after studying the corresponding Fokker-Planck equation. An analysis can be carried on in order to study topological variations of the probability density of the solution process, which are the stochastic counterparts of bifurcations in the deterministic case. For the linear case the deterministic system is the average of the noisy one. In the real nonlinear case this is no longer true, although under weak noise the nonlinear deterministic system behaves approximately as an average of the noisy one (see Feistel and Ebeling [1989] for details). This will be the case in the model presented. See Figure 4 for an illustration.

NUMERICAL MODELLING

According to observations, the typical ratio between the length of the branched portion and that of the trunk in the tree-like structure (see Figures 1 and 3) is 1/3. The tips of the branches are distributed randomly at orthogonal distances of the trunk axis varying from 0 to some 20 cm apart. The overall length of the pattern rarely exceeded 100 cm. Flow speed was estimated in 3 cm/sec.

For numerical experiments the following assumptions are made:

1. The damping term is chosen as $f(x, y) = 1 + \alpha x^2$ where α is a heuristically estimated parameter. A convenient value is $\alpha = 0.1$. For small x the behaviour is approximately linear and the nonlinear effect is negligible. This damping term provides a positive damping decaying with time, in agreement with field data.

2. The natural frequency ω is chosen near the bifurcation limit between the damped oscillatory behaviour and the purely decaying one. For the specific choice of $f(x, y)$ the bifurcation value is $\frac{1}{2}$ and the attraction basin of the origin is the whole phase space.

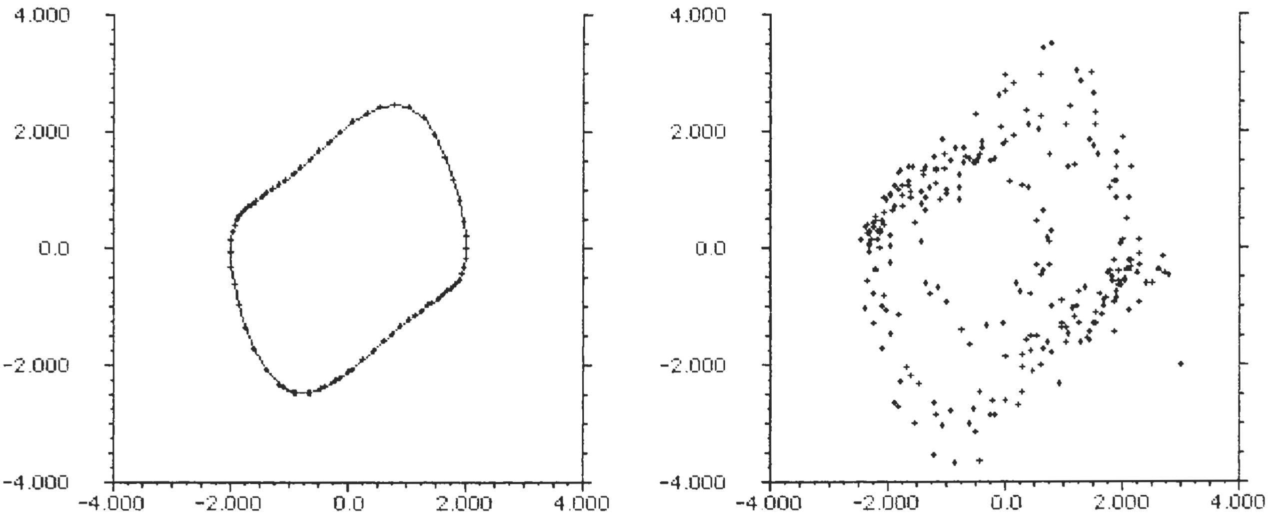


Fig. 4.—Deterministic and noisy Van der Pol oscillators. Observe the deterministic case as averaging the noisy one. A tendency towards bifurcation in this one to a double-peaked density functions is also observed.

3. Initial conditions for x and y are chosen as follows: For $x(0)$ a random value is sampled in a $N(0,15)$ distribution, i.e. upwelling is supposed to take place at locations on a line orthogonal to the trunk. For $y(0)$ any small value, e. g. 0.1 is reasonable, because the upwelling water filament shows a divagating behaviour in the first instants.

4. The noise term is split into two: In the first third of the simulation we allow a larger noise intensity than in the rest. This is consistent with observations, for in the last part of the trajectory, where flow is steadier, only small deviations of the larger central stream are observed, while in the first part (the branches) the weaker streams show a more erratic behaviour. In both cases we take Gaussian white noises with different noise intensities. Typical values are sampled in $N(0,5)$ and $N(0,2)$ distributions.

5. The simulation time was 25 seconds, and the numerical procedure a Runge-Kutta fourth order scheme with a 0.1 mesh interval.

Numerical simulation was carried on with the aid of the modelling package STELLA (Richmond [1987]). This package is designed primarily for building and analyzing models in various scientific areas, but it was used here as a convenient tool for solving (systems of) ordinary differential equations, including those with stochastic terms. The patterns generated by several runs of the model are consistent with observed field data in general shape and dimensions. Compare Figures 1, 3 and 5. A statistical analysis of five samples at the Playa de Maspalomas (Gran Canaria) and several tens of computer runs shows general agreement in trunk/branches ratio and overall length. Thus this modelling scheme is proposed as an auxiliary tool in the study of some interesting geomorphological features.

FINAL COMMENTS AND REMARKS

This modelling technique emphasizes the nonlinear and stochastic aspects of the physical reality, but gives only a rough account of the causal mechanism: These are embodied in the various parameters and require more elaborate analyses.

From the mathematical viewpoint there is an interesting drawback to this modelling scheme: The little water streams start at different points distributed randomly on a two-dimensional domain, a fact that is not captured by our model, where we consider all trajectories starting at points chosen randomly in some line orthogonal to the trunk. Nevertheless, a stochastic two-dimensional modelling could be achieved by considering water upwelling as random in a plane domain and imposing adequate reflecting barriers at the boundaries.

A purely syntactic fractal approach (Crilly 1991) could provide an upstream reconstruction of the pattern. In this view the variable of interest will be the angle of deviation from the common trunk of a particular filament in the stream.

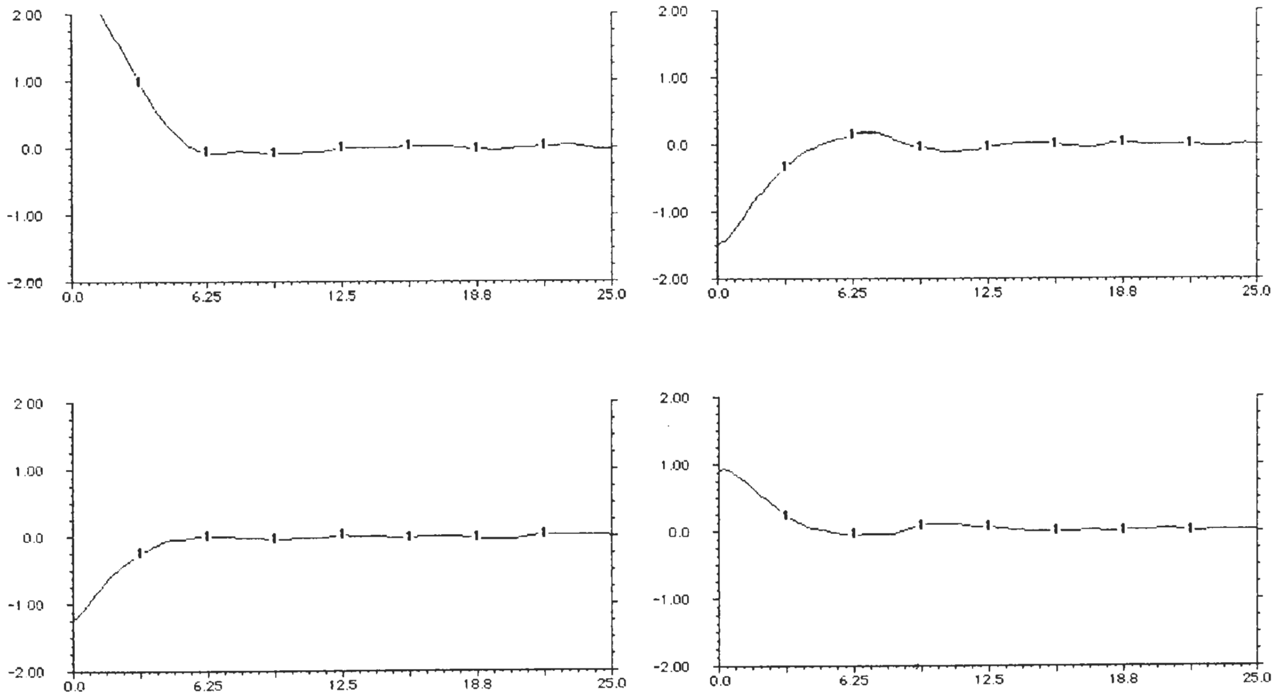


Fig. 5.—Typical realizations of the stochastic process modelling *rill marks*. Time unit is *second* and displacement unit is 10 cms. Compare with Figure 3.

ACKNOWLEDGEMENTS

The author thanks Profs. J. Martínez and J. Mangas of the Departamento de Geología and Prof. A. Santana of the Departamento de Matemáticas at the Facultad de Ciencias del Mar for their valuable suggestions and comments.

REFERENCES

ALLEN, J. R. L. (1986): *Sedimentary Structures, their Character and Physical Basis,* Elsevier, Amsterdam. (Two vols.)

ANDERSON, M. G. (ed.) (1988): *Modelling Geomorphological Systems,* J. Wiley and Sons, Chichester U. K.

CRILLY, J. *et al.* (eds.) (1991): *Fractals and Chaos,* Springer Verlag, New York.

FEISTEL, R.; EBELING, W. (1989): *Evolution of Complex Systems,* Kluwer ed., Dordrecht, The Netherlands.

JORDAN, D.; SMITH, P. (1987): *Nonlinear Ordinary Differential Equations,* Oxford Univ. Press, U.K.

PACHECO, J.; FERNÁNDEZ, I. (1988): «Modelling and computing settling times for suspended particles in the ocean», in: *Schrefler B. and O. Zienkiewicz* (eds.), *Computer Modelling in Ocean Engineering,* 369-374, A. A. Balkema, Rotterdam.

PETTIJOHN, F.; POTTER, P. (1988): *Atlas and Glossary of Primary Sedimentary Structures,* Springer-Verlag, New York.

RICHMOND, B. *et al.* (1987): *An Academic User's Guide to STELLA,* High Performance Systems, Lyme, New York.

Recibido: 10 abril 1992