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The Grade Conjecture and the S_2 Condition

Agustín Marcelo, Félix Marcelo and César Rodríguez

Abstract. Sufficient conditions are given in order to prove the lowest unknown case of the grade conjecture. The proof combines vanishing results of local cohomology and the S_2 condition.

1 Introduction

A classical conjecture due to M. Auslander and included in the so-called homological conjectures (see [2, 5]) is that of the grade, which, in full generality, can be stated as follows: If M is an R-module of finite projective dimension, then $\operatorname{grade} M = \dim R - \dim M$. This conjecture is studied in [4] where it is proved to be a consequence of the vanishing multiplicity conjecture. Using this fact, the authors also proved that the grade conjecture holds for graded modules over graded rings. From the arguments in [5] it follows that the lowest case in which the grade conjecture remains unknown is when $\dim R = 4$, $\operatorname{depth} R = 3$, $\operatorname{grade} M = 2$, $\dim M = 1$, and $\operatorname{depth} M = 0$. In this case the conjecture is unknown to hold even for cyclic modules. The purpose of the present paper is to give sufficient conditions in order to prove that the grade conjecture holds true in this case. To do this we use the following type of matrices: Let A be an $m \times n$ matrix with coefficients in a ring R with rows $R_1, \dots, R_m$ and columns $C_1, \dots, C_n$. The matrix A is said to be *initial* if the quotient modules $R^n/\langle R_1, \dots, R_m\rangle$, $R^m/\langle C_1, \dots, C_n\rangle$ are torsion free. We remark that this definition is self-dual; *i.e.*, A is an initial matrix if and only if A^t is.

The main result can then be stated as follows:

Theorem *Let R be a Noetherian local domain of dimension 4 and depth 3 satisfying the S_2 condition, let M be a finitely-generated R module of dimension 1, projective dimension 3 and depth 0, and let*

$$0 \longrightarrow F_3 \xrightarrow{\varphi_3} F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \longrightarrow M \longrightarrow 0$$

be a finite-rank free resolution of M. Assume the following two conditions hold:

(a) $M_{\mathfrak{p}} = 0$ *for every* $\mathfrak{p} \in \operatorname{Spec} R$ *with* $\operatorname{height} \mathfrak{p} = 1$.
(b) *The matrix of φ_2 is initial.*

Then, the grade conjecture holds true for M.

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Remark We recall that a finitely generated R-module M satisfies the Serre's condition S_n if $\operatorname{depth} M_{\mathfrak{p}} \geq \min(n, \dim R_{\mathfrak{p}})$ for all $\mathfrak{p} \in \operatorname{Spec} R$. We also note that we are not assuming that R is an equidimensional ring, so that condition (a) above is not redundant.

2 A Basic Lemma

Lemma *Let* $0 \xrightarrow{\varphi} U \to V \to T \to 0$ *be an exact sequence of finitely generated* R*-modules, in which* U *satisfies the* S_2 *condition,* V *satisfies the* S_1*, and* $T_{\mathfrak{p}} = 0$ *for every* $\mathfrak{p} \in \operatorname{Spec} R$ *with* height $\mathfrak{p} = 1$. *Then,* φ *is an isomorphism; i.e.,* $T = 0$.

Proof Set $\mathfrak{a} = \operatorname{Ann} T$. As $T_{\mathfrak{p}} = 0$ for every $\mathfrak{p}$ with height $\mathfrak{p} \leq 1$, and $\operatorname{Supp}(R/\mathfrak{a}) = \operatorname{Supp}(T)$, we have height $\mathfrak{a} \geq 2$. Next, taking into account that (*e.g.*, see [3, 15.G])

$$\operatorname{depth}_{\mathfrak{a}} U = \inf_{\mathfrak{a} \subseteq \mathfrak{p}} (\operatorname{depth} U_{\mathfrak{p}}),$$

and that $\dim R_{\mathfrak{p}} \geq 2$, we obtain

$$\operatorname{depth}_{\mathfrak{a}} U \geq \inf_{\mathfrak{a} \subseteq \mathfrak{p}} \big(2, \dim(R_{\mathfrak{p}})\big) \geq 2,$$

as U satisfies the S_2 condition. Moreover, by applying the cohomological interpretation of depth (*e.g.*, see [1, 3.4, p. 217]),

$$\operatorname{depth}_{\mathfrak{a}} U \geq s \Leftrightarrow H^i_{\mathfrak{a}}(U) = 0, \quad i < s,$$

it turns out that $H^i_{\mathfrak{a}}(U) = 0$, $i = 0, 1$. Similarly, it is not difficult to prove that $H^0_{\mathfrak{a}}(V) = 0$. Finally, as $H^0_{\mathfrak{a}}(T) = T$, we conclude by using the exact sequence

$$0 = H^0_{\mathfrak{a}}(V) \longrightarrow H^0_{\mathfrak{a}}(T) \longrightarrow H^1_{\mathfrak{a}}(U) = 0. \qquad \blacksquare$$

3 Initial Matrices

In this section we establish some basic facts that we need in the proof of the theorem.

Proposition 1 *Let* M *be a finitely generated* R*-module and let* $F \xrightarrow{\varphi} G \to M \to 0$ *be a free presentation of* M*. Assume that the matrix of* φ *in some bases of* F, G*, respectively, is initial. Then* M *is a torsion-free module.*

Proof This follows from the very definition of an initial matrix given above. $\blacksquare$

Proposition 2 *Let* R *be a local domain satisfying the* S_2 *condition and let* $F \xrightarrow{\varphi} G \to M \to 0$ *be as in Proposition 1. Then* $\operatorname{Im} \varphi$ *satisfies the* S_2 *condition.*

Proof We must show that depth $\operatorname{Im}\varphi_{\mathfrak{p}} \geq \inf(2, \dim R_{\mathfrak{p}})$.

(i) Assume height $\mathfrak{p} = 1$. As $\operatorname{Im}\varphi$ is torsion free (for it is a submodule of a free module), taking into account that R satisfies S_2, it is easy to see that depth $\operatorname{Im}\varphi_{\mathfrak{p}} = 1$.

(ii) Assume height $\mathfrak{p} \geq 2$. Let us consider the exact sequence

$$0 \longrightarrow \operatorname{Im}\varphi_{\mathfrak{p}} \longrightarrow G_{\mathfrak{p}} \longrightarrow M_{\mathfrak{p}} \longrightarrow 0.$$

By Proposition 1, the module $M_{\mathfrak{p}}$ is torsion free. Hence depth $M_{\mathfrak{p}} \geq 1$, and we have $H^0_{\mathfrak{p}}(M_{\mathfrak{p}}) = 0$. We also have $H^1_{\mathfrak{p}}(G_{\mathfrak{p}}) = 0$. Therefore $H^1_{\mathfrak{p}}\big(\operatorname{Im}(\varphi_{\mathfrak{p}})\big) = 0$, thus yielding depth $\operatorname{Im}\varphi_{\mathfrak{p}} \geq 2$. Thus depth $\operatorname{Im}\varphi_{\mathfrak{p}} \geq \inf(2, \dim R_{\mathfrak{p}})$, and $\operatorname{Im}\varphi$ satisfies the S_2 condition. ■

4 Proof of the Theorem

We first remark that, according to the Auslander-Buchsbaum formula, there always exists a free resolution of M as that in the statement of the theorem. We proceed in three steps:

(i) $\operatorname{Hom}_R(M, R) = 0$,
(ii) $\operatorname{Ext}^1_R(M, R) = 0$,
(iii) $\operatorname{Ext}^2_R(M, R) = 0$.

(i) $\operatorname{Hom}_R(M, R) = 0$. By dualizing the epimorphism $F_0 \to M \to 0$ we obtain an injection $0 \to M^\vee \to F_0^\vee$, where $M^\vee$ is a torsion R-module since $(M^\vee)_{\mathfrak{p}} = (M_{\mathfrak{p}})^\vee = 0$ if height $\mathfrak{p} \leq 1$. As $F_0^\vee$ is torsion free we also have $M^\vee = 0$.

(ii) $\operatorname{Ext}^1_R(M, R) = 0$. By applying (i), and dualizing the short exact sequence $0 \to \operatorname{Im}\varphi_1 \to F_0 \to M \to 0$, we obtain an exact sequence

$$0 \longrightarrow F_0^\vee \longrightarrow (\operatorname{Im}\varphi_1)^\vee \longrightarrow \operatorname{Ext}^1_R(M, R) \longrightarrow 0.$$

Now $F_0^\vee$ satisfies the S_2 condition and $(\operatorname{Im}\varphi_1)^\vee$ satisfies the S_1 condition as this module is a submodule of $F_1^\vee$. Finally $\operatorname{Ext}^1_R(M, R)_{\mathfrak{p}} = 0$ if height $\mathfrak{p} = 1$, since $M_{\mathfrak{p}} = 0$. From the previous lemma we deduce $\operatorname{Ext}^1_R(M, R) = 0$.

(iii) $\operatorname{Ext}^2_R(M, R) = 0$. By dualizing the complex of free R-modules

$$0 \longrightarrow F_3 \xrightarrow{\varphi_3} F_2 \xrightarrow{\varphi_2} F_1,$$

we obtain

$$F_1^\vee \xrightarrow{\varphi_2^\vee} F_2^\vee \xrightarrow{\varphi_3^\vee} F_3^\vee.$$

Hence we have an exact sequence

$$0 \longrightarrow \operatorname{Im}(\varphi_2^\vee) \longrightarrow \ker(\varphi_3^\vee) \longrightarrow \operatorname{Ext}^2_R(M, R) \longrightarrow 0.$$

By virtue of the hypotheses, the matrix of $\varphi_2^\vee$ is initial since it is the transpose map of φ_2. Now, taking into account Proposition 2 it follows that $\mathrm{Im}(\varphi_2^\vee)$ satisfies the S_2 condition and $\ker \varphi_3^\vee$ satisfies the S_1 condition for it is a submodule of $F_2^\vee$. Moreover, since $M_\mathfrak{p} = 0$ it follows that $\mathrm{Ext}_R^2(M, R)_\mathfrak{p} = 0$ if height $\mathfrak{p} \leq 1$. The desired result thus follows from the lemma.

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Departamento de Matemáticas
Universidad de Las Palmas de Gran Canaria
Campus de Tafira
35017 Las Palmas de Gran Canaria
Spain
email: amarcelo@dma.ulpgc.es
wernitz@idecnet.com
cesar@dma.ulpgc.es